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The horn-feed problem: sound waves in a tube joined to a cone, and related problems

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Abstract A semi-infinite tube is joined to a semi-infinite cone. Waves propagating in the tube towards the join are partly reflected and partly radiated into the cone. The problem is to determine these wave fields. Two modal expansions are used, one in the tube and one in the cone. However, their regions of convergence do not overlap: there is a region $\mathcal{D}$ near the join where neither expansion converges. It is shown that the expansions can be connected by judicious applications of Green's theorem in $\mathcal{D}$. The resulting equations are solved asymptotically, for long waves or for narrow cones. Related two-dimensional problems are also solved. Applications to acoustics, electromagnetics and hydrodynamics are considered.

Keywords Horns · Matched eigenfunction expansions · Waveguide junctions

1 Introduction

Suppose that we have a semi-infinite tube that is open at one end. The tube is rigid and has a circular cross-section. If a sound wave propagates within the tube towards the end, it will be partly reflected back along the tube and partly radiated out into the surrounding fluid. The problem of calculating the reflected and radiated fields can be solved exactly, using the Wiener–Hopf technique; see [1] or [2, Sect. 3.4].

Suppose now that the tube is joined to a circular cone, a *conical flange* or *horn*; the tube and the cone have a common centreline. As before, a wave propagates in the tube towards the tube–cone junction, and the problem is to calculate the reflected and the radiated fields: we call this the *horn-feed problem*. (Note that it is assumed here that there is no mean flow in the tube–cone structure.)

The horn-feed problem arises in various forms. As posed, it suggests a connection with idealised musical instruments. Indeed, brass instruments have been modelled as joined conical frusta [3].

Electromagnetic horns have also been studied extensively. This work has been reviewed by Risser [4], Love [5], Olver et al. [6] and Bird and Love [7]; see also [8] and [9, Sect. 5.24]. For high-frequency approximations, see [10] and [11, Sects. 8.6, 8.7].

There are also analogous two-dimensional problems, where a waveguide joins a wedge-shaped region. Let us describe such a problem in the context of small-amplitude water waves (see Sect. 3 for details). Thus, suppose that

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a semi-infinite channel is joined to a wedge-shaped ocean; both have rigid vertical walls, a common centreline, and are filled with water of constant depth, h. An incident wave from one region is partly reflected back and partly transmitted into the other region. For water waves, the most interesting problem is when the incident wave comes from the ocean: we call this the *ocean-inlet problem*. However, in the context of acoustics or electromagnetism, the opposite problem is most important: a channel mode is incident, and one wants to calculate the field transmitted into the wedge. We also call this a horn-feed problem.

In this paper, we begin by describing a semi-analytical method for solving these two-dimensional problems. Our method is a generalization of the method of matched eigenfunction expansions (MEE). Thus, we use modal expansions in the channel and in the wedge. However, there is a region at the junction between the channel and the wedge in which neither expansion is valid; the region is a segment of a circle, and is denoted by $\mathcal{D}$ below. In standard applications of MEE, either $\mathcal{D}$ is absent or a third expansion is used in $\mathcal{D}$. Here, we connect the two expansions across $\mathcal{D}$ using certain applications of Green's theorem in $\mathcal{D}$.

Our method leads to infinite systems of linear algebraic equations. It could be developed into a numerical method for solving horn-feed problems by appropriate truncation of the linear systems, but that is not the focus of this paper. (For some remarks on possible numerical implementations, see Sect. 9.) Instead, we use the method to obtain various analytical approximations, for long waves or narrow wedges (Sect. 6). Having the option to truncate the linear system at different levels means that approximations can be refined: we are not restricted to one-dimensional approximations based on Webster-like ordinary differential equations (as described in Sect. 2).

Three-dimensional problems are treated in Sect. 8. Specifically, we consider axisymmetric waves in the tube–cone geometry described earlier. (The axisymmetry constraint could be removed at the expense of using more complicated special functions.) Again, low-frequency and narrow-cone approximations are extracted. Further applications and generalizations are expected.

2 Previous work

The two-dimensional problems described in Sect. 1 can be reduced to solving the Helmholtz equation, $(\nabla^2+k^2)u = 0$, in a domain

$$-\infty < x < \infty, \quad -b_1(x) < y < b_2(x),$$

where x and y are Cartesian coordinates, and the functions b_1 and b_2 are given; specifically, for the channel–wedge geometry, we have

$$b_1(x) = b_2(x) = \begin{cases} b, & x \le 0, \\ b + x\tan\alpha, & x > 0, \end{cases} \tag{1}$$

where b and α are positive constants with $0 < \alpha < \frac{1}{2}\pi$ (Fig. 1). Although we shall give a method for solving the corresponding boundary-value problems exactly, it is of interest to consider various approximate methods.

One approach assumes that the breadth $B(x) = b_1(x) + b_2(x)$ varies slowly with x and that the waves are such that

$$u(x, y) \simeq U(x),$$

so that u does not vary significantly in the lateral direction. It follows that U satisfies

$$\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}x}\left(B\frac{\mathrm{d}U}{\mathrm{d}x}\right) + k^2U = 0, \tag{2}$$

an ordinary differential equation known as *Webster's horn equation* [12]; for derivations and generalizations, see [13], [14, p. 360] and [15].

Strictly, Webster's horn equation is not valid when $B(x)$ has discontinuities in slope, as with (1). Nevertheless, substitution of (1) for $x > 0$ in (2) gives

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + \frac{1}{x + b\cot\alpha}\frac{\mathrm{d}U}{\mathrm{d}x} + k^2U = 0, \tag{3}$$

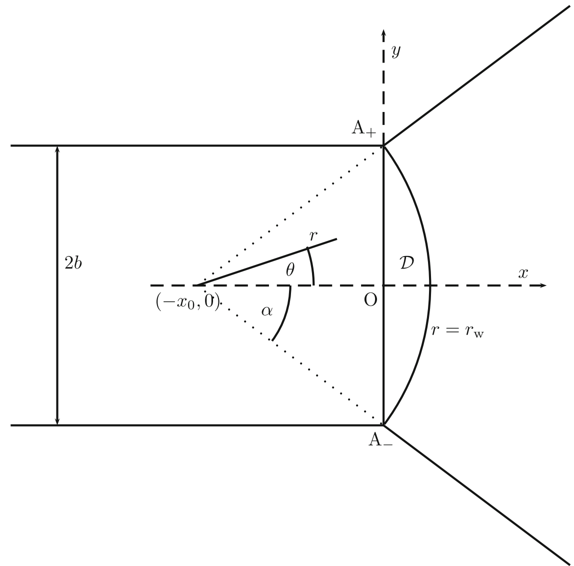

Fig. 1 The geometry of the horn-feed problem. There are two basic geometrical parameters, b and α. We also use $x_0 = b \cot\alpha$, $r_\mathrm{w} = b \csc\alpha$ and dimensionless wave-numbers, $K = kx_0$ and $\kappa = kr_\mathrm{w}$

which is Bessel's equation of order zero with $x + b\cot\alpha$ as the independent variable [16, p. 7]. Thus, with an assumed time dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$, the outgoing solution of (3) is

$$U(x) = c_\mathrm{w} H_0^{(1)}(k[x + x_0]), \quad x > 0,$$

where c_w is a constant, $x_0 = b\cot\alpha$ and $H_0^{(1)}$ is a Hankel function. For $x < 0$, we suppose that there is a wave incident upon the junction; hence, as B is constant,

$$U(x) = \mathrm{e}^{\mathrm{i}kx} + R_\mathrm{w}\mathrm{e}^{-\mathrm{i}kx}, \quad x < 0,$$

where R_w is another constant, the reflection coefficient. Continuity of U and U' across $x = 0$ gives

$$1 + R_\mathrm{w} = c_\mathrm{w} H_0^{(1)}(kx_0) \quad \text{and} \quad 1 - R_\mathrm{w} = \mathrm{i}c_\mathrm{w} H_1^{(1)}(kx_0),$$

using $H_0' = -H_1$. Hence

$$R_\mathrm{w} = \frac{H_0^{(1)}(K) - \mathrm{i}H_1^{(1)}(K)}{H_0^{(1)}(K) + \mathrm{i}H_1^{(1)}(K)} \quad \text{and} \quad c_\mathrm{w} = \frac{2}{H_0^{(1)}(K) + \mathrm{i}H_1^{(1)}(K)}, \tag{4}$$

where $K = kx_0$. We shall return to this solution in Sect. 6.

Equation 2 can be used for long waves on water of constant finite depth h. However, it is then customary to use shallow-water theory, leading to

$$\frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}x}\left(B\frac{\mathrm{d}U}{\mathrm{d}x}\right) + \frac{\omega^2}{gh}U = 0, \tag{5}$$

where ω is the circular frequency and g is the acceleration due to gravity. Equation 5 is discussed by Lamb [17, Sect. 186] and by LeBlond and Mysak [18, Eq. 28.28]. Applications of (5) to piecewise-linear $B(x)$ were made by Dean [16].

In the electromagnetic context, with the geometry (1), the waveguide $x < 0$ is joined to the *sectoral horn* $x > 0$ at the *throat* $x = 0$. Usually, the incident wave travels along the waveguide towards the throat, and the frequency is chosen below the first cut-off so that only the fundamental duct mode can propagate in the waveguide. Thus, the incident wave does not vary in the lateral direction. One can also determine modes in the horn; the outgoing modes are defined by (10) below. In particular, the fundamental mode is simply $H_0^{(1)}(kr)$, giving a cylindrical mode emanating from $r = 0$, a point on the centreline at $x = -x_0 = -b\cot\alpha$. For small values of α (so that the wedge is "narrow"), one might argue that an incident fundamental duct mode is transmitted as a fundamental outgoing wedge mode, with zero reflection. It was clearly recognised by Risser [4, pp. 354, 357] that this is an over-simplification in many situations, but no rigorous solution was available so as to quantify the error. A similar remark was made much later by Jones [9, p. 275]; see also the discussion of this and related problems by Lewin [19] and by Olver et al. [6, Chap. 4].

Given two (infinite) modal expansions, one in the duct and one in the wedge, the problem is to relate them across the throat region $\mathcal{D}$. One could *assume* that both expansions are valid on $x = 0$ or on $r = r_{\mathrm{w}} = b \csc \alpha$, and then match as usual. One could set up a boundary-integral equation around the boundary of $\mathcal{D}$. Another possibility is to make an approximation in $\mathcal{D}$. For example, one might suppose that, in $\mathcal{D}$, the Helmholtz equation can be replaced by Laplace's equation (implying that $kb \ll 1$). This is essentially a form of matched asymptotic expansions. This approach was used by Chester [20] to analyse the problem of acoustic waves in a circular tube meeting a conical horn, and, as he notes, generalizes some calculations of Lord Rayleigh [21, Sect. 313]. Later, Chester [22] gave another analysis for small α; he assumed that the modal expansion in the cone was convergent on $x = 0$, where he matched to the modal expansion in the tube.

3 Formulation

Let $Oxyz$ be Cartesian coordinates, so that $z = 0$ is the mean free surface and $z = -h$ is the rigid bottom. We consider a symmetric configuration in which a semi-infinite channel is joined to a wedge-shaped ocean. The channel has walls at $y = \pm b$, $x < 0$. The wedge has walls at $y = \pm(b + x \tan \alpha)$, $x > 0$. Thus, the channel has width $2b$, the wedge has angle 2α, and the positive x-axis points into the wedge; see Fig. 1. The channel walls meet the wedge walls at $(x, y) = (0, \pm b)$; denote these points by $\mathrm{A}_\pm$. We assume that $0 < \alpha < \frac{1}{2}\pi$; $\alpha = 0$ corresponds to an infinite parallel-walled channel whereas $\alpha = \frac{1}{2}\pi$ corresponds to the ocean-inlet problem solved by Dalrymple and Martin [23].

Throughout the water, the total potential can be expressed as

$$\mathfrak{Re}\left\{ u(x, y) \frac{\cosh k(h + z)}{\cosh kh} \, \mathrm{e}^{-\mathrm{i}\omega t} \right\}, \tag{6}$$

where k is the unique positive root of the dispersion relation $\omega^2 = gk \tanh kh$. Thus, the potential (6) satisfies the boundary conditions on the bottom and on the free surface. It also satisfies the three-dimensional Laplace equation if u solves the two-dimensional Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0. \tag{7}$$

In addition, u must have a vanishing normal derivative,

$$\frac{\partial u}{\partial n} = 0, \tag{8}$$

on the channel and wedge walls, and it will have to satisfy certain conditions at infinity; these will be specified later when we have chosen the incident field.

Introduce plane polar coordinates (r, θ) centred at the tip of the wedge, $(x, y) = (-b \cot \alpha,\ 0)$. Thus,

$$x + b \cot \alpha = r \cos \theta \quad \text{and} \quad y = r \sin \theta,$$

so that the wedge walls are given by $\theta = \pm\alpha$. In the region

$$\mathcal{D}_{\mathrm{w}} \equiv \{(x, y) : r > r_{\mathrm{w}} \equiv b \csc \alpha,\ |\theta| < \alpha\},$$

we can write the total potential $u \equiv u_{\mathrm{w}}$ as

$$u_{\mathrm{w}} = \sum_{n=0}^{\infty} \epsilon_n c_n \psi_n + \tilde{u}_{\mathrm{w}} \tag{9}$$

where $\epsilon_0 = 1$, $\epsilon_n = 2$ for $n > 0$,

$$\psi_n = H^{(1)}_{\nu_n}(kr)\,\cos\nu_n\theta \quad \text{with } \nu_n = n\pi/\alpha, \tag{10}$$

and $H^{(1)}_\nu$ is a Hankel function. The potential $\tilde{u}_\mathrm{w}$ will be prescribed later (depending on the incident field) and the coefficients c_n, $n = 0, 1, 2, \ldots$, are to be found. Each *wedge mode* ψ_n satisfies (7) within the wedge, (8) on the wedge walls, and is outgoing as $r \to \infty$.

In the channel

$$\mathcal{D}_\mathrm{c} \equiv \{(x, y) : x < 0,\ |y| < b\},$$

we can write $u \equiv u_\mathrm{c}$ as

$$u_\mathrm{c} = \sum_{n=0}^{\infty} \epsilon_n a_n \chi_n + \tilde{u}_\mathrm{c} \tag{11}$$

where

$$\chi_n = \mathrm{e}^{-\mathrm{i}\beta_n x}\cos\lambda_n y \quad \text{with } \lambda_n = n\pi/b$$

and

$$\beta_n = \begin{cases} \sqrt{k^2 - \lambda_n^2}, & 0 \le \lambda_n \le k, \\ \mathrm{i}\sqrt{\lambda_n^2 - k^2}, & \lambda_n > k. \end{cases} \tag{12}$$

The potential $\tilde{u}_\mathrm{c}$ will be prescribed later and the coefficients a_n, $n = 0, 1, 2, \ldots$, are to be found. Each *duct mode* χ_n satisfies (7) within the channel and (8) on the channel walls. Each χ_n is an outgoing propagating wave as $x \to -\infty$, or it decays exponentially as $x \to -\infty$.

For simplicity, we have assumed that all fields are symmetric about the centreline $y = 0$. This is not an essential requirement; we could include antisymmetric wedge and duct modes, if needed.

The complete region filled with fluid is $\mathcal{D}_\mathrm{w} \cup \mathcal{D}_\mathrm{c} \cup \mathcal{D}$, where $\mathcal{D}$ is a segment of a circle, defined by

$$\mathcal{D} \equiv \{(x, y) : x > 0,\ r < r_\mathrm{w} \equiv b\csc\alpha\},$$

with corners at $\mathrm{A}_\pm$. These corners imply that we cannot use either modal expansion in $\mathcal{D}$.

It remains to determine the coefficients a_n and c_n by relating the two modal expansions across $\mathcal{D}$. In previous applications of such expansion methods, the geometry has been such that either $\mathcal{D}$ is absent (for example, two channels of different widths) or a third modal expansion is used in $\mathcal{D}$. Here, we note that the boundary of $\mathcal{D}$, $\partial\mathcal{D}$, comprises two pieces, $\partial\mathcal{D} = \partial\mathcal{D}_\mathrm{w} \cup \partial\mathcal{D}_\mathrm{c}$, where $\partial\mathcal{D}_\mathrm{w}$ is the circular arc $r = r_\mathrm{w}$, $|\theta| < \alpha$, and $\partial\mathcal{D}_\mathrm{c}$ is the line segment $x = 0$, $|y| < b$. We can use the modal expansions (9) and (11) on $\partial\mathcal{D}_\mathrm{w}$ and $\partial\mathcal{D}_\mathrm{c}$, respectively. The crucial observation for the success of our method (described in Sect. 4) is that $\partial\mathcal{D}$ does not include any pieces of the walls; see the discussion at the end of Sect. 4. Thus, the method will fail if the channel and wedge do not have a common centreline.

4 The matching method

Let φ and Φ be two non-singular solutions of (7) in $\mathcal{D}$. Then, an application of Green's theorem in $\mathcal{D}$ to φ and Φ gives

$$\int_{\partial\mathcal{D}} \left(\varphi\,\frac{\partial\Phi}{\partial n} - \Phi\,\frac{\partial\varphi}{\partial n}\right)\mathrm{d}s = 0.$$

Taking the outward normal, we obtain

$$\mathcal{A}(\varphi, \Phi) = \mathcal{B}(\varphi, \Phi) \tag{13}$$

where

$$\mathcal{A}(\varphi, \Phi) = \int_{-b}^{b} \left(\varphi \frac{\partial \Phi}{\partial x} - \Phi \frac{\partial \varphi}{\partial x} \right)_{x=0} \mathrm{d}y,$$

$$\mathcal{B}(\varphi, \Phi) = \int_{-\alpha}^{\alpha} \left(\varphi \frac{\partial \Phi}{\partial r} - \Phi \frac{\partial \varphi}{\partial r} \right)_{r=r_{\mathrm{w}}} r_{\mathrm{w}} \, \mathrm{d}\theta.$$

If we replace φ in (13) by the total potential u, we can then use (11) in $\mathcal{A}$ and (9) in $\mathcal{B}$ to give

$$\sum_{n=0}^{\infty} \epsilon_n a_n A_n(\Phi) + \sum_{n=0}^{\infty} \epsilon_n c_n B_n(\Phi) = F(\Phi) \tag{14}$$

where

$$A_n(\Phi) = \mathcal{A}(\chi_n, \Phi) = -\mathcal{A}(\Phi, \chi_n),$$
$$B_n(\Phi) = -\mathcal{B}(\psi_n, \Phi) = \mathcal{B}(\Phi, \psi_n),$$
$$F(\Phi) = \mathcal{B}(\tilde{u}_{\mathrm{w}}, \Phi) - \mathcal{A}(\tilde{u}_{\mathrm{c}}, \Phi).$$

Equation 14 holds for all admissible Φ, that is, for all regular solutions of the Helmholtz equation in $\mathcal{D}$. We make two choices for Φ,

$$\Phi = \mathrm{e}^{+\mathrm{i}\beta_m x} \cos \lambda_m y = \chi_m^* \quad \text{and} \quad \Phi = H_{\nu_m}^{(2)}(kr) \cos \nu_m \theta = \psi_m^*,$$

where $m \geq 0$ is an integer. These equations define χ_m^* and ψ_m^*, respectively. Note that ψ_m^* is the complex conjugate of ψ_m for all m. However, χ_m^* is the complex conjugate of χ_m only for propagating modes, where β_m is real.

Orthogonality of $\{\cos \lambda_n y\}$ over $|y| \leq b$ implies that

$$\epsilon_n A_n(\chi_m^*) = 4\mathrm{i}b\beta_m \delta_{mn},$$

where δ_{ij} is the Kronecker delta. Similarly, orthogonality of $\{\cos \nu_n \theta\}$ over $|\theta| \leq \alpha$, together with the Wronskian for Bessel functions, gives

$$\epsilon_n B_n(\psi_m^*) = 8\mathrm{i}(\alpha/\pi)\delta_{mn}.$$

Thus, (14) gives

$$4\mathrm{i}b\beta_m \, a_m + \sum_{n=0}^{\infty} \epsilon_n B_n(\chi_m^*) \, c_n = F(\chi_m^*), \tag{15}$$

$$8\mathrm{i}(\alpha/\pi) \, c_m + \sum_{n=0}^{\infty} \epsilon_n A_n(\psi_m^*) \, a_n = F(\psi_m^*) \tag{16}$$

for $m = 0, 1, 2, \ldots$, which is an infinite system of linear algebraic equations for a_n and c_n, $n = 0, 1, 2, \ldots$.

In general, the remaining integrals in (15) and (16) must be evaluated numerically. Note that, from (13), we have

$$A_n(\psi_m^*) = \mathcal{A}(\chi_n, \psi_m^*) = \mathcal{B}(\chi_n, \psi_m^*),$$
$$B_n(\chi_m^*) = \mathcal{B}(\chi_m^*, \psi_n) = \mathcal{A}(\chi_m^*, \psi_n),$$

so that we can write *all* integrals over the circular arc $\partial\mathcal{D}_{\mathrm{w}}$ or over the line segment $\partial\mathcal{D}_{\mathrm{c}}$, as convenient.

In the calculations above, we made two choices for Φ, namely χ_m^* and ψ_m^*. Other choices could be made. For example, as $A_n(\psi_m) = 0$, ψ_m^* could be replaced by $2J_{\nu_m}(kr) \cos \nu_m \theta$ in (16).

Notice that if $\partial\mathcal{D}$ had included a piece of the walls (as would have happened if the channel and wedge did not have a common centreline), we would have been obliged to use functions Φ that satisfy $\partial\Phi/\partial n = 0$ on that piece of the walls. Although such functions are easily constructed, we would lose orthogonality over other pieces of $\partial\mathcal{D}$.

5 The two-dimensional horn-feed problem

For this problem, the incident field is a duct mode. The simplest problem is when the fundamental mode is chosen, whence

$$\tilde{u}_{\mathrm{w}} \equiv 0 \quad \text{and} \quad \tilde{u}_{\mathrm{c}} = \mathrm{e}^{\mathrm{i}kx} = \chi_0^*.$$

Thus

$$F(\Phi) = -\mathcal{A}(\tilde{u}_{\mathrm{c}}, \Phi) = -\mathcal{B}(\tilde{u}_{\mathrm{c}}, \Phi).$$

In particular, $F(\chi_m^*) = \mathcal{A}(\chi_m^*, \chi_0^*) = 0$.

Let us make a statement on energy conservation for the horn-feed problem. Thus, an application of Green's theorem gives

$$\int_{-b}^{b} \left(u_{\mathrm{c}} \frac{\partial \overline{u_{\mathrm{c}}}}{\partial x} - \overline{u_{\mathrm{c}}} \frac{\partial u_{\mathrm{c}}}{\partial x} \right) \mathrm{d}y = \int_{-\alpha}^{\alpha} \left(u_{\mathrm{w}} \frac{\partial \overline{u_{\mathrm{w}}}}{\partial r} - \overline{u_{\mathrm{w}}} \frac{\partial u_{\mathrm{w}}}{\partial r} \right) r \, \mathrm{d}\theta, \tag{17}$$

where the overbar denotes complex conjugation. Evaluate the left-hand side of (17) at any negative value of x, using the expansion (11) (with $\tilde{u}_{\mathrm{c}} = \mathrm{e}^{\mathrm{i}kx}$) and orthogonality of $\{\cos \lambda_n y\}$. Evaluate the right-hand side of (17) at any value of $r > r_{\mathrm{w}}$, using the expansion (9) (with $\tilde{u}_{\mathrm{w}} = 0$), orthogonality of $\{\cos \nu_n \theta\}$ and the Wronskian for Bessel functions. The result is

$$1 - \sum_{n=0}^{N_{\mathrm{p}}-1} \epsilon_n \frac{\beta_n}{k} |a_n|^2 = \frac{2\alpha}{\pi k b} \sum_{n=0}^{\infty} \epsilon_n |c_n|^2, \tag{18}$$

where N_{p} is the number of propagating duct modes. The identity (18) expresses energy conservation for the horn-feed problem.

6 The horn-feed problem: approximations

Suppose we have one reflected propagating mode in the channel (so that $kb < \pi$), we neglect the evanescent modes in the channel and we take just one cylindrical mode in the wedge. Thus, we write

$$u_{\mathrm{c}} = \mathrm{e}^{\mathrm{i}kx} + R\mathrm{e}^{-\mathrm{i}kx}, \quad u_{\mathrm{w}} = c_0 H_0^{(1)}(kr), \tag{19}$$

where $R \equiv a_0$ is the reflection coefficient. Energy conservation, (18), reduces to

$$1 - |R|^2 = \frac{2\alpha}{\pi k b} |c_0|^2. \tag{20}$$

Taking $m = 0$ in (15) and (16) gives

$$4\mathrm{i}kbR + c_0 B_0(\chi_0^*) = 0, \quad 8\mathrm{i}(\alpha/\pi)c_0 + RA_0(\psi_0^*) = F(\psi_0^*),$$

whence

$$R = \overline{A_0(\psi_0^*)} \frac{F(\psi_0^*)}{\Delta}, \quad c_0 = -4\mathrm{i}kb \frac{F(\psi_0^*)}{\Delta}, \quad \Delta = 32kb\frac{\alpha}{\pi} + |A_0(\psi_0^*)|^2, \tag{21}$$

where we have used

$$B_0(\chi_0^*) = \mathcal{A}(\chi_0^*, \psi_0) = \overline{\mathcal{A}(\chi_0, \psi_0^*)} = \overline{A_0(\psi_0^*)}.$$

6.1 Fixed α, small kb

For fixed geometry and long waves, we can approximate further. As α is fixed, it is convenient to write the coefficients in (21) in terms of integrals at $r = r_\mathrm{w}$ across the wedge. Thus, as $\chi_0 = \mathrm{e}^{-\mathrm{i}k(r\cos\theta - x_0)}$,

$$A_0(\psi_0^*) = \mathcal{B}(\chi_0, \psi_0^*) = \mathrm{e}^{\mathrm{i}kx_0} \int_{-\alpha}^{\alpha} \mathrm{e}^{-\mathrm{i}kr_\mathrm{w}\cos\theta} \left(\frac{\partial \psi_0^*}{\partial r} + \mathrm{i}k\psi_0^* \cos\theta \right)_{r=r_\mathrm{w}} r_\mathrm{w}\,\mathrm{d}\theta$$
$$= -\kappa \left(H_1^{(2)}(\kappa)\,\overline{\mathcal{E}(\kappa)} + H_0^{(2)}(\kappa)\,\overline{\mathcal{E}'(\kappa)} \right) \mathrm{e}^{\mathrm{i}\kappa\cos\alpha},$$
$$F(\psi_0^*) = -\mathcal{B}(\chi_0^*, \psi_0^*) = \kappa \left(H_1^{(2)}(\kappa)\,\mathcal{E}(\kappa) + H_0^{(2)}(\kappa)\,\mathcal{E}'(\kappa) \right) \mathrm{e}^{-\mathrm{i}\kappa\cos\alpha},$$

where $\kappa = kr_\mathrm{w} = kb\csc\alpha$, $x_0 = b\cot\alpha = r_\mathrm{w}\cos\alpha$ is the distance from the wedge tip to the throat and

$$\mathcal{E}(\kappa) = \int_{-\alpha}^{\alpha} \mathrm{e}^{\mathrm{i}\kappa\cos\theta}\,\mathrm{d}\theta.$$

As α is fixed and kb is small, $\kappa = kr_\mathrm{w} \ll 1$. Therefore, we can use standard small-argument approximations for the Hankel functions, $H_0^{(2)}(\kappa) = -(2\mathrm{i}/\pi)\log\kappa + O(1)$ and $H_1^{(2)}(\kappa) = (2\mathrm{i}/\pi)\kappa^{-1} + O(\kappa\log\kappa)$, together with $\mathcal{E}(\kappa) = 2\alpha + O(\kappa)$, $\mathcal{E}'(\kappa) = 2\mathrm{i}\sin\alpha + O(\kappa)$ and $\mathrm{e}^{\pm\mathrm{i}\kappa\cos\alpha} = 1 + O(\kappa)$ as $\kappa \to 0$. Hence, as $\kappa\sin\alpha = kb$,

$$\overline{A_0(\psi_0^*)} \sim F(\psi_0^*) \sim 4\mathrm{i}(\alpha/\pi)\{1 - \mathrm{i}(kb/\alpha)\log\kappa\}.$$

Thus, $\Delta \sim (4\alpha/\pi)^2\{1 + 2kb\pi/\alpha\}$,

$$R \sim -1 + 2\mathrm{i}(kb/\alpha)\log\kappa \quad \text{and} \quad c_0 \sim \pi(kb/\alpha)\{1 - \mathrm{i}(kb/\alpha)\log\kappa\}.$$

The expression for R shows that, in the limit $kb \to 0$, the incident wave in the duct is reflected as if the duct were closed with $u = 0$ on $x = 0$. The next term is reminiscent of the known exact solution (obtained by the Wiener–Hopf technique) for an open-ended channel (formally, put $\alpha = \pi$). Indeed, if we suppose that kb is small in that exact solution, we find that the reflection coefficient is given, asymptotically, by

$$R_\pi \sim -1 + 2\mathrm{i}(kb/\pi)\log(kb) \quad \text{as } kb \to 0;$$

see [24, p. 28] or [2, p. 110].

We can also compare R and c_0 with the corresponding results obtained by solving Webster's horn equation, R_w and c_w. Thus, using small-argument approximations for the Hankel functions in (4), we obtain

$$R_\mathrm{w} \sim -1 + 2\mathrm{i}kx_0\log(kx_0) \quad \text{and} \quad c_\mathrm{w} \sim \pi kx_0\{1 - \mathrm{i}kx_0\log(kx_0)\}.$$

Thus, all three estimates compare well.

Note that if $|R|$ is of interest, it can be calculated readily from (20) and the estimate for c_0.

6.2 Small α, fixed kb

Evidently, the approximations obtained in Sect. 6.1 are inappropriate for small α. In this case, it is convenient to write the coefficients in (21) in terms of integrals at $x = 0$ across the throat. Thus,

$$A_0(\psi_0^*) = \mathcal{A}(\chi_0, \psi_0^*) = \int_{-b}^{b} \left(\frac{\partial \psi_0^*}{\partial x} + \mathrm{i}k\psi_0^* \right)_{x=0} \mathrm{d}y,$$
$$F(\psi_0^*) = -\mathcal{A}(\chi_0^*, \psi_0^*) = -\int_{-b}^{b} \left(\frac{\partial \psi_0^*}{\partial x} - \mathrm{i}k\psi_0^* \right)_{x=0} \mathrm{d}y.$$

We have $\partial f(r)/\partial x = f'(r)\cos\theta$ and

$$r = r_0(y) = \sqrt{x_0^2 + y^2} \quad \text{and} \quad \cos\theta = x_0/r_0(y) \quad \text{on } x = 0.$$

Thus,

$$\frac{1}{k}\left(\frac{\partial\psi_0^*}{\partial x} \pm \mathrm{i}k\psi_0^*\right)_{x=0} = -\frac{x_0}{r_0}H_1^{(2)}(kr_0) \pm \mathrm{i}H_0^{(2)}(kr_0).$$

Suppose now that α is small so that x_0/b is large. We use standard large-argument asymptotic approximations for the Hankel functions,

$$H_0^{(2)}(z) \sim \left(1 + \frac{\mathrm{i}}{8z}\right)E(z), \quad H_1^{(2)}(z) \sim \left(\mathrm{i} + \frac{3}{8z}\right)E(z),$$

with $E(z) = \sqrt{2/(\pi z)}\exp\{-\mathrm{i}(z - \pi/4)\}$. To justify using these approximations, $z = kr_0 \sim kx_0$ must be large, implying that kb must be bounded away from zero. Then,

$$\left(\frac{\partial\psi_0^*}{\partial x} - \mathrm{i}k\psi_0^*\right)_{x=0} \sim -2\mathrm{i}kE(kx_0), \quad \left(\frac{\partial\psi_0^*}{\partial x} + \mathrm{i}k\psi_0^*\right)_{x=0} \sim -\frac{E(kx_0)}{2x_0}.$$

Hence $F(\psi_0^*) \sim 4\mathrm{i}kbE(kx_0)$ and $A(\psi_0^*) \sim -(b/x_0)E(kx_0)$. Then, as $x_0 \simeq b/\alpha$ for small α, we obtain $\Delta \sim 32kb\alpha/\pi$,

$$R \sim -\frac{\mathrm{i}\alpha}{4kb} \quad \text{and} \quad c_0 \sim \frac{\pi kb}{2\alpha}E(kx_0). \tag{22}$$

Thus, $|c_0|^2 \sim \frac{1}{2}\pi kb/\alpha$ so that energy is conserved to leading order: the right-hand side of (20) approaches unity as $\alpha \to 0$ whereas $|R|^2 = O(\alpha^2)$.

The corresponding estimates based on (4) are

$$R_\mathrm{w} \sim -\frac{\mathrm{i}\tan\alpha}{4kb} \quad \text{and} \quad c_\mathrm{w} \sim \frac{\pi kb}{2\tan\alpha}E(kx_0).$$

These agree with (22) as $\alpha \to 0$; the estimates for the reflection coefficient also agree with those found by Rice [25], Leonard and Yen [26] and Riblet [27].

The wave transmitted into the wedge is

$$c_0 H_0^{(1)}(kr) \sim \alpha^{-1}\sqrt{(b/x_0)(b/r)}\,\mathrm{e}^{\mathrm{i}k(r-x_0)} \sim \mathrm{e}^{\mathrm{i}kx} \quad \text{as } \alpha \to 0.$$

This is as expected: when $\alpha \to 0$, the wedge reduces to a continuation of the channel, implying zero reflection and the unchanged propagation of the incident wave.

6.3 Refined approximations

One merit of our approach is that it can be used to develop improved approximations. For example, suppose we augment the approximation (19) with an additional mode in the wedge:

$$u_\mathrm{c} = \mathrm{e}^{\mathrm{i}kx} + \tilde{R}\mathrm{e}^{-\mathrm{i}kx}, \quad u_\mathrm{w} = \tilde{c}_0 H_0^{(1)}(kr) + 2\tilde{c}_1 H_{\nu_1}^{(1)}(kr)\cos\nu_1\theta. \tag{23}$$

Here, $\nu_1 = \pi/\alpha$. Equations 15 and 16 give

$$4\mathrm{i}kb\tilde{R} + \tilde{c}_0 B_0 + 2\tilde{c}_1 B_1 = 0,$$

$$8\mathrm{i}(\alpha/\pi)\tilde{c}_0 + \tilde{R}A_{00} = F_0,$$

$$8\mathrm{i}(\alpha/\pi)\tilde{c}_1 + \tilde{R}A_{01} = F_1.$$

Here, we have used a shorthand notation: $A_{00} = A_0(\psi_0^*)$, $A_{01} = A_0(\psi_1^*)$, $B_0 = B_0(\chi_0^*)$, $B_1 = B_1(\chi_0^*)$, $F_0 = F(\psi_0^*)$ and $F_1 = F(\psi_1^*)$. Solving gives

$$\tilde{R} = \frac{R\Delta + 2B_1F_1}{\Delta + 2B_1A_{01}}, \qquad \frac{8\mathrm{i}\alpha}{\pi}\tilde{c}_1 = \frac{F_1\Delta - A_{01}B_0F_0}{\Delta + 2B_1A_{01}},$$

$$\frac{8\mathrm{i}\alpha}{\pi}\tilde{c}_0 = \frac{8\mathrm{i}(\alpha/\pi)c_0\Delta + 2B_1(F_0A_{01} - F_1A_{00})}{\Delta + 2B_1A_{01}},$$

with R, c_0 and Δ defined by (21). In particular,

$$\frac{\tilde{c}_1}{\tilde{c}_0} = \frac{F_1\Delta - A_{01}B_0F_0}{8\mathrm{i}(\alpha/\pi)c_0\Delta + 2B_1(F_0A_{01} - F_1A_{00})}. \tag{24}$$

Let us estimate this ratio for small kb and fixed α. From results in Sect. 6.1, we have $\Delta \sim \gamma^2$, $F_0 \sim B_0 \sim -A_{00} \sim \mathrm{i}\gamma$ and $c_0 \sim 4(\kappa/\gamma)\sin\alpha$ with $\gamma = 4\alpha/\pi$ and $\kappa = kr_\mathrm{w}$ Hence, (24) reduces to

$$\frac{\tilde{c}_1}{\tilde{c}_0} \sim \frac{\gamma(F_1 + A_{01})}{8\mathrm{i}\kappa\gamma\sin\alpha + 2\mathrm{i}B_1(F_1 + A_{01})}. \tag{25}$$

Next, we find that

$$A_{01} = \mathcal{B}(\chi_0, \psi_1^*) = \kappa\left(\Lambda'(\kappa)\overline{\mathcal{E}_1(\kappa)} - \Lambda(\kappa)\overline{\mathcal{E}_1'(\kappa)}\right)\mathrm{e}^{\mathrm{i}\kappa\cos\alpha},$$

$$F_1 = -\mathcal{B}(\chi_0^*, \psi_1^*) = -\kappa\left\{\Lambda'(\kappa)\mathcal{E}_1(\kappa) - \Lambda(\kappa)\mathcal{E}_1'(\kappa)\right\}\mathrm{e}^{-\mathrm{i}\kappa\cos\alpha}$$

and $B_1 = \overline{A_{01}}$, where

$$\Lambda(\kappa) = H^{(2)}_{\nu_1}(\kappa) \quad \text{and} \quad \mathcal{E}_1(\kappa) = \int_{-\alpha}^{\alpha} \mathrm{e}^{\mathrm{i}\kappa\cos\theta}\cos\nu_1\theta\,\mathrm{d}\theta.$$

Then, we use $\Lambda(\kappa) \sim \mathrm{i}\Lambda_0\kappa^{-\nu_1}$ and $\mathcal{E}_1(\kappa) \sim \mathrm{i}\mathcal{E}_0\kappa$ as $\kappa \to 0$, where $\Lambda_0 = \pi^{-1}2^{\nu_1}\Gamma(\nu_1)$ and $\mathcal{E}_0 = 2(\nu_1^2 - 1)^{-1}\sin\alpha$ are both real. Hence, $F_1 \sim (\nu_1 + 1)\mathcal{E}_0\Lambda_0\kappa^{1-\nu_1}$ as $\kappa \to 0$, with exactly the same estimate holding for both A_{01} and B_1. Then, inspection of the denominator in (25) shows that the first term is $O(\kappa)$ and the second is $O(\kappa^{2-2\nu_1})$; the second is dominant because $\nu_1 = \pi/\alpha > 1$. Hence, (25) reduces further to

$$\frac{\tilde{c}_1}{\tilde{c}_0} \sim \frac{\gamma}{2\mathrm{i}B_1} \sim \frac{\gamma\,\kappa^{\nu_1-1}}{2\mathrm{i}(\nu_1 + 1)\mathcal{E}_0\Lambda_0}; \tag{26}$$

this justifies neglecting $\tilde{c}_1$ compared to $\tilde{c}_0$ when $\kappa \to 0$.

7 The ocean-inlet problem

For this problem, the incident field is $u_\mathrm{inc} = \mathrm{e}^{-\mathrm{i}kx} = \chi_0$. We assume that, *in the absence of the channel*, the total potential in the wedge-shaped ocean is given by $u_\mathrm{inc} + u_\mathrm{ref}$, where u_ref is the known reflected field. The calculation of u_ref is a classical problem, first solved by Macdonald in 1902. The relevant formulae and references are given by Bowman and Senior [28, Sect. 6.2.2]. Thus, one can write the total potential as a contour integral (which is convenient for large r) or as an eigenfunction expansion. For the latter, we set $\nu = 2\alpha/\pi$, $\Omega = \pi - \alpha$, $\rho = r$, $\phi = \pi - \theta$ and $\phi_0 = \pi$ in [28, Eq. 6.40] to give

$$u_\mathrm{inc} + u_\mathrm{ref} = \frac{\pi}{\alpha}\sum_{n=0}^{\infty}\epsilon_n\,\mathrm{e}^{-\mathrm{i}\pi\nu_n/2}\hat{\psi}_n$$

where

$$\hat{\psi}_n = J_{\nu_n}(kr)\cos\nu_n\theta \quad \text{with } \nu_n = n\pi/\alpha,$$

is a *regular wedge mode* (it is the real part of the outgoing wedge mode ψ_n, defined by (10)). When $\alpha = \pi/N$, with N an integer, the solution simplifies. For example,

$$u_\mathrm{ref} = \mathrm{e}^{\mathrm{i}kx} + 2\cos ky \qquad \text{when } \alpha = \tfrac{1}{4}\pi \text{ and}$$
$$u_\mathrm{ref} = 2\exp\left(\tfrac{1}{2}\mathrm{i}kx\right)\cos\left(\tfrac{1}{2}\sqrt{3}ky\right) \qquad \text{when } \alpha = \tfrac{1}{3}\pi.$$

Now, when the channel is present, we take

$$\tilde{u}_\mathrm{w} = u_\mathrm{inc} + u_\mathrm{ref} \quad \text{and} \quad \tilde{u}_\mathrm{c} \equiv 0,$$

whence $F(\Phi) = \mathcal{B}(\tilde{u}_\mathrm{w}, \Phi) = \mathcal{A}(\tilde{u}_\mathrm{w}, \Phi)$. In particular, making use of the Wronskian for Bessel functions gives

$$F(\psi_m^*) = -4\mathrm{i}\exp\{-\tfrac{1}{2}\mathrm{i}\pi\nu_m\},$$

whereas the expression for $F(\chi_m^*)$ does not simplify (except for special choices of α).

In principle, we can now solve the ocean-inlet problem, using (15) and (16), and we can develop approximate solutions as in Sect. 6.

8 The three-dimensional horn-feed problem

We consider a circular tube joined to a cone. The geometry is axisymmetric: we take the x-axis as the axis of symmetry. Introduce cylindrical polar coordinates, ϱ, ϕ and x. The tube is $\varrho = b$, $x < 0$. The cone is $\varrho = b + x\tan\alpha$, $x > 0$. A wave propagates in the tube towards the join at $x = 0$, and the problem is to find the reflected and radiated fields.

For simplicity, we assume that the incident wave, u_{inc}, is axisymmetric and so the reflected and radiated fields are also axisymmetric: there is no dependence on the angle ϕ.

In the tube ($x < 0$, $0 \le \varrho < b$), we can write

$$u = u_{\mathrm{inc}} + \sum_{n=0}^{\infty} a_n \chi_n \quad \text{with } \chi_n = \mathrm{e}^{-\mathrm{i}\beta_n x} J_0(\lambda_n \varrho), \tag{27}$$

where β_n is defined by (12), $\lambda_0 = 0$ and $\lambda_n b = j_{1,n}$, the nth positive zero of the Bessel function J_1. In particular, $\chi_0 = \mathrm{e}^{-\mathrm{i}kx}$. Each mode χ_n is an axisymmetric solution of the three-dimensional Helmholtz equation; each mode satisfies $\partial\chi_n/\partial\varrho = 0$ on the wall $\varrho = b$; and each mode either propagates towards $x = -\infty$ or it decays exponentially as $x \to -\infty$. We also have orthogonality [29, Eq. 11.4.5]:

$$\int_0^b J_0(\lambda_m \varrho) J_0(\lambda_n \varrho)\varrho\, \mathrm{d}\varrho = \frac{1}{2} b^2 J_0^2(\lambda_n b)\delta_{mn}. \tag{28}$$

In the cone, we use spherical polar coordinates, r and θ, with $x + b\cot\alpha = r\cos\theta$, so that the cone's surface is at $\theta = \alpha$. Then, inside the cone ($r > r_{\mathrm{w}} = b\csc\alpha$, $0 \le \theta < \alpha$), we can write [30]

$$u = \sum_{n=0}^{\infty} c_n \psi_n \quad \text{with } \psi_n = h_{\nu_n}^{(1)}(kr) P_{\nu_n}(\cos\theta), \tag{29}$$

where $h_\nu^{(1)}(z) = \sqrt{\pi/(2z)}\, H_{\nu+1/2}^{(1)}(z)$, $P_\nu(z)$ is a Legendre function and the (real) quantities ν_n are defined by $P'_{\nu_n}(\cos\alpha) = 0$. In particular, $\nu_0 = 0$ and $\psi_0 = h_0^{(1)}(kr) = \mathrm{e}^{\mathrm{i}kr}/(\mathrm{i}kr)$. Each mode ψ_n is an axisymmetric solution of the three-dimensional Helmholtz equation; each mode satisfies $\partial\psi_n/\partial\theta = 0$ on the wall $\theta = \alpha$; and each mode gives an outgoing wave as $r \to \infty$. We also have orthogonality [30, Eq. 18.267]:

$$\int_0^\alpha P_{\nu_m}(\cos\theta) P_{\nu_n}(\cos\theta)\sin\theta\, \mathrm{d}\theta = p_n \delta_{mn}, \tag{30}$$

with $p_0 = 1 - \cos\alpha$ and

$$p_n = -\frac{\sin\alpha}{2\nu_n + 1} \left. \frac{\partial^2 P_q(\cos\alpha)}{\partial q\, \partial\alpha} \right|_{q=\nu_n} P_{\nu_n}(\cos\alpha).$$

Let us now adapt the matching method of Sect. 4 to the axisymmetric horn-feed problem. First, Green's theorem in the throat region $\mathcal{D}$ (bounded by the disc $\varrho < b$ at $x = 0$ and the spherical cap, $r = r_{\mathrm{w}}$, $0 \le \theta < \alpha$) gives (13) with $\mathcal{A}$ and $\mathcal{B}$ defined by

$$\mathcal{A}(\varphi, \Phi) = \frac{1}{b}\int_0^b \left(\varphi \frac{\partial\Phi}{\partial x} - \Phi\frac{\partial\varphi}{\partial x} \right)_{x=0} \varrho\, \mathrm{d}\varrho,$$

$$\mathcal{B}(\varphi, \Phi) = \frac{1}{b}\int_0^\alpha \left(\varphi \frac{\partial\Phi}{\partial r} - \Phi\frac{\partial\varphi}{\partial r} \right)_{r=r_{\mathrm{w}}} r_{\mathrm{w}}^2 \sin\theta\, \mathrm{d}\theta.$$

Next, we replace φ in (13) by the total potential u, using (27) in $\mathcal{A}$ and (29) in $\mathcal{B}$ to give

$$\sum_{n=0}^{\infty} a_n A_n(\Phi) + \sum_{n=0}^{\infty} c_n B_n(\Phi) = F(\Phi) \tag{31}$$

where $A_n(\Phi) = \mathcal{A}(\chi_n, \Phi)$, $B_n(\Phi) = -\mathcal{B}(\psi_n, \Phi)$ and $F(\Phi) = \mathcal{A}(\Phi, u_{\mathrm{inc}})$. Equation 31 holds for all regular solutions Φ of the Helmholtz equation in $\mathcal{D}$. As in Sect. 4, we make two choices for Φ,

$$\Phi = \mathrm{e}^{+\mathrm{i}\beta_m x} J_0(\lambda_m \varrho) = \chi_m^* \quad \text{and} \quad \Phi = \overline{\psi_m} = \psi_m^*,$$

where $m \geq 0$ is an integer. These equations define χ_m^* and ψ_m^*. Using these in (31), together with the orthogonality relations, (28) and (30), and the Wronskian for $h_\nu^{(1)}$, gives

$$\mathrm{i}b\beta_m J_0^2(\lambda_m b)\, a_m + \sum_{n=0}^{\infty} B_n(\chi_m^*)\, c_n = F(\chi_m^*), \tag{32}$$

$$\frac{2\mathrm{i}}{kb} p_m\, c_m + \sum_{n=0}^{\infty} A_n(\psi_m^*)\, a_n = F(\psi_m^*) \tag{33}$$

for $m = 0, 1, 2, \ldots$, which is an infinite system of linear algebraic equations for a_n and c_n, $n = 0, 1, 2, \ldots$. Also, for the simplest horn-feed problem, we have $u_{\mathrm{inc}} = \mathrm{e}^{\mathrm{i}kx} = \chi_0^*$, whence $F(\chi_m^*) = 0$.

Let us approximate, as in Sect. 6. Thus, write $u = \mathrm{e}^{\mathrm{i}kx} + R\mathrm{e}^{-\mathrm{i}kx}$ in the tube (so that $a_0 = R$) and $u = c_0 h_0^{(1)}(kr)$ in the cone. Then, taking $m = 0$ in (32) and (33) gives

$$\mathrm{i}kbR + c_0 B_0(\chi_0^*) = 0, \quad [2\mathrm{i}/(kb)]p_0 c_0 + RA_0(\psi_0^*) = F(\psi_0^*),$$

which can be solved for R and c_0. For example, as $B_0(\chi_0^*) = \overline{A_0(\psi_0^*)}$,

$$R = F(\psi_0^*)\overline{A_0(\psi_0^*)} \left\{2p_0 + |A_0(\psi_0^*)|^2\right\}^{-1}.$$

Unlike in two dimensions, the integrals here can be evaluated exactly. We find that

$$A_0(\psi_0^*) = \mathcal{B}(\chi_0, \psi_0^*) = \frac{p_0}{\mathrm{i}kb}\mathrm{e}^{-\mathrm{i}\kappa},$$

$$F(\psi_0^*) = \mathcal{B}(\psi_0^*, \chi_0^*) = \frac{\mathrm{e}^{-\mathrm{i}\kappa}}{\mathrm{i}kb}\{1 + \cos\alpha - 2\mathrm{e}^{\mathrm{i}\kappa p_0}\},$$

where $\kappa = kr_{\mathrm{w}} = kb\csc\alpha$ and $p_0 = 1 - \cos\alpha$. Hence,

$$R = \frac{1 + \cos\alpha - 2\mathrm{e}^{\mathrm{i}\kappa p_0}}{p_0 + 2(kb)^2}. \tag{34}$$

For fixed α and small kb, (34) gives

$$R \sim -1 - 2\mathrm{i}kb\csc\alpha. \tag{35}$$

On the other hand, for small α and fixed kb, $p_0 \sim \alpha^2/2$, $\kappa p_0 \sim kb\alpha/2$ and (34) gives

$$R \sim -\frac{\mathrm{i}\alpha}{2kb}. \tag{36}$$

Let us compare these results with those obtained by solving Webster's horn equation. For a horn of cross-sectional area $S(x)$, this equation is $U'' + (S'/S)U' + k^2U = 0$. Within the cone, $S(x) = \pi(x + x_0)^2$, so the outgoing solution is $U(x) = c_{\mathrm{w}} h_0^{(1)}(k[x + x_0])$ [12]. In the tube, $U(x) = \mathrm{e}^{\mathrm{i}kx} + R_{\mathrm{w}}\mathrm{e}^{-\mathrm{i}kx}$. Continuity conditions at $x = 0$ then give

$$R_{\mathrm{w}} = \frac{h_0^{(1)}(K) - \mathrm{i}h_1^{(1)}(K)}{h_0^{(1)}(K) + \mathrm{i}h_1^{(1)}(K)} = \frac{1}{2\mathrm{i}K - 1},$$

where $K = kx_0 = \kappa\cos\alpha$. We have $R_{\mathrm{w}} \sim -1 - 2\mathrm{i}kb\cot\alpha$ as $K \to 0$, which should be compared with (35). Similarly, $R_{\mathrm{w}} \sim (2\mathrm{i}kb)^{-1}\tan\alpha$ as $K \to \infty$, which compares well with the small-α approximation (36).

9 Discussion and conclusion

We have described a method for treating problems where two modal expansions are connected via a region in which neither expansion is valid. Such problems arise in several contexts: one is the horn-feed problem, where a waveguide (tube or channel) is connected to a horn (cone or wedge). The method leads to infinite systems of linear algebraic equations for the modal coefficients. Truncated systems were solved analytically: good agreement with various low-frequency and narrow-horn approximations was found.

It is known that the infinite systems arising from matched modal expansions should be truncated with care. This was first shown by Mittra [31] for the problem of a bifurcated waveguide, with an infinite channel, $|y| < b$, containing a thin semi-infinite screen along $y = 0$, $x > 0$; see also [32, Sect. 2-3]. Mittra showed that it is necessary to take proper account of the edge condition at the tip of the screen. In our context, this means the behaviour of u near the points $A_\pm$ in Fig. 1, where $u \sim \{x^2 + (y \mp b)^2\}^{\pi/(2\pi+2\alpha)}$. Our linear systems were obtained by making two choices for the function Φ in (14), but, as already noted, other choices could have been made. It is anticipated that this flexibility could be exploited in order to incorporate the edge conditions, perhaps by adapting Porter's method (see, for example, [33, Sect. 5]). Related issues of linear independence and conditioning have not yet been investigated.

References

1. Levine H, Schwinger J (1948) On the radiation of sound from an unflanged circular pipe. Phys Rev 73:383–406
2. Noble B (1988) Methods based on the Wiener–Hopf technique. Chelsea, New York
3. Caussé R, Kergomard J, Lurton X (1984) Input impedance of brass musical instruments—comparison between experiment and numerical models. J Acoust Soc Am 75:241–254
4. Risser JR (1949) Waveguide and horn feeds. In: Silver S (ed) Microwave antenna theory and design. McGraw-Hill, New York pp 334–387
5. Love AW (ed) (1976) Electromagnetic horn antennas. IEEE Press, New York
6. Olver AD, Clarricoats PJB, Kishk AA, Shafai L (1994) Microwave horns and feeds. IEE, London
7. Bird TS, Love AW (2007) Horn antennas. In: Volakis JL (ed) Antenna engineering handbook, 4th edn. McGraw-Hill, New York pp 14-1–14-74
8. Green HE (2006) The radiation pattern of a conical horn. J Electromagn Waves Appl 20:1149–1160
9. Jones DS (1986) Acoustic and electromagnetic waves. Oxford University Press, Oxford
10. Kaloshin VA (2009) Scattering matrix for a junction of two horns. Russ J Math Phys 16:246–259
11. Borovikov VA, Kinber BY (1994) Geometrical theory of diffraction. Institution of Electrical Engineers, London
12. Webster AG (1919) Acoustical impedance, and the theory of horns and of the phonograph. Proc Natl Acad Sci USA 5:275–282
13. Benade AH, Jansson EV (1974) On plane and spherical waves in horns with nonuniform flare I. Theory of radiation, resonance frequencies, and mode conversion. Acustica 31:79–98
14. Pierce AD (1989) Acoustics. Acoustical Society of America, New York
15. Martin PA (2004) On Webster's horn equation and some generalizations. J Acoust Soc Am 116:1381–1388
16. Dean RG (1964) Long wave modification by linear transitions. J Waterw Harb Div Proc ASCE 90:1–29
17. Lamb H (1932) Hydrodynamics, 6th edn. Cambridge University Press, Cambridge
18. LeBlond PH, Mysak LA (1978) Waves in the ocean. Elsevier, Amsterdam
19. Lewin L (1970) On the inadequacy of discrete mode-matching techniques in some waveguide discontinuity problems. IEEE Trans Microw Theory Tech MTT 18:364–369
20. Chester W (1983) The acoustic impedance of a semi-infinite tube fitted with a conical flange. Z Angew Math Phys 34:412–417
21. Rayleigh (1896) The theory of sound, vol 2. Reprinted, Dover, New York, 1945
22. Chester W (1987) The acoustic impedance of a semi-infinite tube fitted with a conical flange: Part II. J Sound Vib 116:371–377
23. Dalrymple RA, Martin PA (1996) Water waves incident on an infinitely long rectangular inlet. Appl Ocean Res 18:1–11
24. Chester W (1950) The propagation of sound waves in an open-ended channel. Phil Mag 41(7):11–33
25. Rice SO (1949) A set of second-order differential equations associated with reflections in rectangular wave guides—application to guide connected to horn. Bell Syst Tech J 28:136–156
26. Leonard DJ, Yen JL (1957) Junction of smooth flared wave guides. J Appl Phys 28:1441–1448
27. Riblet HJ (1977) An alternate derivation of Lewin's formula. IEEE Trans Microw Theory Tech MTT 25:711–712
28. Bowman JJ, Senior TBA (1969) The wedge. In: Bowman JJ, Senior TBA, Uslenghi PLE (eds) Electromagnetic and acoustic scattering by simple shapes. North-Holland, Amsterdam pp 252–283
29. Abramowitz M, Stegun IA (eds) (1965) Handbook of mathematical functions. Dover, New York

30. Bowman JJ (1969) The cone. In: Bowman JJ, Senior TBA, Uslenghi PLE (eds) Electromagnetic and acoustic scattering by simple shapes. North-Holland, Amsterdam pp 637–701
31. Mittra R (1963) Relative convergence of the solution of a doubly infinite set of equations. J Res Natl Bur Stand 67D:245–254
32. Mittra R, Lee SW (1971) Analytical techniques in the theory of guided waves. Macmillan, New York
33. Porter R, Porter D (2000) Water wave scattering by a step of arbitrary profile. J Fluid Mech 411:131–164