Geometrical representation of the fundamental mode of a Gaussian beam in oblate spheroidal coordinates

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A new geometrical model for the fundamental mode of a Gaussian beam is presented in the oblate spheroidal coordinate system. The model is an interpretation of a Gaussian amplitude wave function, which is an exact solution of the scalar Helmholtz equation. The model uses the skew-line generator of a hyperboloid of one sheet as a raylike element on a contour of constant amplitude. The geometrical characteristics of the skew line and the consequences of treating it as a ray are explored in depth. Finally, the skew line is used to build a nonorthogonal coordinate system that permits straight-line propagation of a Gaussian beam in three-dimensional space.

INTRODUCTION

In the past, depictions of the fundamental mode of a propagating Gaussian beam consisted of both mathematical descriptions of the wave function and geometrical interpretations of these descriptions. The mathematical model used most frequently is the one introduced by Kogelnik and Li,[1] which is only an approximate solution of the scalar wave equation and is expressed in the Cartesian coordinate system. Other authors[2–4] have used the oblate spheroidal coordinate system, shown in Fig. 1, to express the propagation of a Gaussian beam because of the simplicity of modeling a contour of constant amplitude in the beam as a hyperboloid of one sheet, which is one of the oblate spheroidal coordinate surfaces. It is also possible to obtain exact solutions to the Helmholtz equation in this coordinate system, as was shown by Einziger and Raz[3] and Landesman and Barrett.[4] In the latter paper, the mathematical model of a Gaussian beam was extended to include an entire family of analytic solutions to the wave equation that possess an amplitude distribution that is fundamentally Gaussian. This is the mathematical model that is used here to develop a geometrical depiction of the fundamental mode of a Gaussian beam.

Since Gaussian beams are used in a wide variety of optical systems and instruments, a geometrical model of the beam is an absolute necessity for system design and analysis. Ideally, the model should correspond to the mathematical description and interpret it in the light of geometrical optics. Many methods exist for predicting the first-order properties of a Gaussian beam as it traverses an optical system. Those that are wedded to the mathematical description of Kogelnik and Li suffer from the approximations inherent in that wave function as well as from the awkwardness of describing the beam in the Cartesian coordinate system. A more useful, and currently more popular, method is that introduced by Arnaud,[5–7] which unites the geometrical constructs of the oblate spheroidal coordinate system with the wave fronts and amplitude contours of the model of Kogelnik and Li. In particular, Arnaud used the geometrical concept of a ruled surface that is produced by the motion of a skew line. A ruled surface is a surface generated by the motion of a straight line, called a rectilinear generator, in three-dimensional space. A hyperboloid of revolution of one sheet, η = constant in the oblate spheroidal coordinate system, is one example of a ruled surface, which, in this case, results from the rotation of a straight line about an axis that it does not cut. The straight line is therefore skewed to the axis of symmetry, the z axis, and is referred to as a skew line. The skew line lies on the surface of the hyperboloid and is everywhere tangent to it.

The skew line has enjoyed some popularity as the basis for a Gaussian beam model because of the simplicity and the well-behaved nature of a straight line. Arnaud treated a complex representation of the skew line as a complex ray that obeys the laws of geometrical optics. Further work has since extended the representation of the fundamental mode of a Gaussian beam by complex rays,[8–10] a concept that Felsen[11] vigorously disputed. In contrast, a real representation of the skew line leads to an elegant design tool for predicting the first-order properties of a Gaussian beam in an optical system. This method was introduced by Shack[12] and later was developed more fully by Kessler and Shack.[13]

All these constructs hinge on a geometrical interpretation of the traditional description of the fundamental mode of a Gaussian beam, as given by Kogelnick and Li.[1] This mathematical model is written as

$$u(x, y, z) = \frac{w_0}{w(z)} \exp\left\{-i\left[kz - \Phi + \frac{k(x^2 + y^2)}{2R(z)}\right] - \frac{x^2 + y^2}{w^2(z)}\right\}, \tag{1}$$

where

$$\Phi = \tan^{-1} \frac{z}{z_0}, \tag{2}$$

$$R(z) = \frac{z^2 + {z_0}^2}{z}, \tag{3}$$

and

$$w(z) = w_0\left[1 + \left(\frac{z}{z_0}\right)^2\right]^{1/2}. \tag{4}$$

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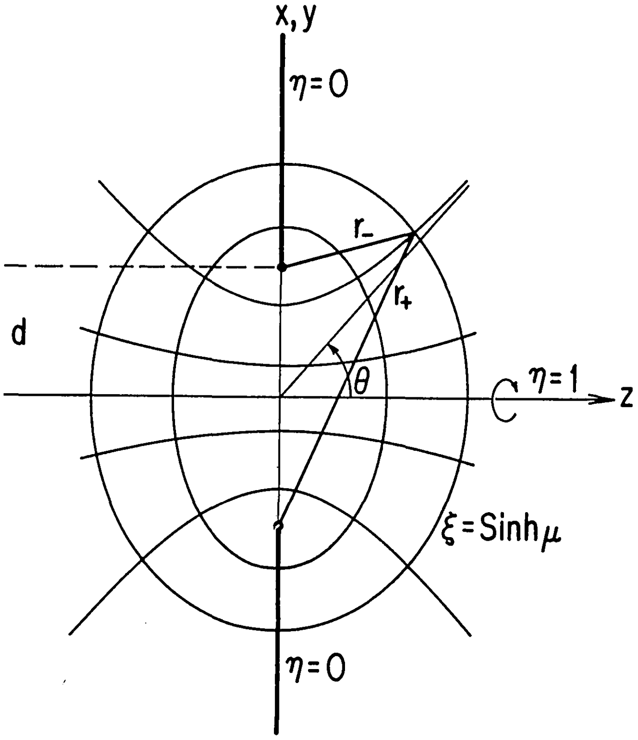

Fig. 1. Oblate spheroidal coordinate system.

The beam parameters z_0 and w_0 are known as the Rayleigh range and the beam waist, respectively, and are related by

$$z_0 = \frac{kw_0^2}{2}, \tag{5}$$

with k the propagation constant of the medium. The wave function described by Eq. (1) is interpreted as having a spherical wave front, with a radius of curvature $R(z)$ and a Gaussian amplitude distribution that expands according to the hyperbola defined by Eq. (4). The factor $w(z)$ represents the distance from the axis at which the field amplitude is $1/e$ times that on the axis. The phase difference Φ is interpreted as the phase difference between the Gaussian beam and an ideal plane wave, and the amplitude factor w_0/w gives the expected on-axis intensity decrease attributable to expansion of the beam. The disadvantages of this mathematical model include its reliance on the Cartesian coordinate system, the limitation of its validity to only the paraxial region, and the fact that it is only an approximate solution of the approximated scalar Helmholtz equation.

The geometrical model developed here uses a real representation of the skew line to interpret the new mathematical description of a Gaussian beam given by Landesman and Barrett.[4] This beam is the zero-order member of a new family of analytic solutions to the scalar wave equation and is given by

$$\psi_{00}(\xi, \eta, \phi) = \frac{\exp(ikd\xi)\exp[-kd(1-\eta)]\exp(-i\tan^{-1}\xi/\eta)}{kd(\xi^2+\eta^2)^{1/2}}, \tag{6}$$

where ξ, η, and ϕ are the oblate spheroidal coordinates, k is the propagation constant of the medium, and $2d$ is the focus spacing in the oblate spheroidal coordinate system (described in detail in Appendix A). The function $\psi_{00}(\xi, \eta, \phi)$ describes a wave with a wave front that is nominally a section of an oblate ellipsoid (ξ = constant). This wave front is modified by the term $\exp(-i\tan^{-1}\xi/\eta)$ as η varies from 1 to 0. The exponential amplitude $\exp[-kd(1-\eta)]$ specifies the amplitude distribution on the oblate ellipsoid. In the paraxial limit, this term reduces to the traditional Gaussian amplitude distribution, as shown in detail by Landesman and Barrett.[4] Further, the amplitude factor $[\eta^2 + \xi^2]^{-1/2}$ ensures that the wave energy falls to zero in the limit as the wave propagates to infinity; that is, as $z \to \infty$, $\xi \to \infty$, and $\psi_{00}(\xi, \eta, \phi) \to 0$. Finally, the exponential time dependence, $\exp(-i\omega t)$, is understood and has been suppressed.

This new beam description possesses significant advantages over previous attempts to model a Gaussian beam, including its simplicity and the elimination of the paraxial approximation. Since the wave function given by Eq. (6) is not confined to the paraxial region, it can be used to model beams with large divergence angles. Furthermore, this new wave function predicts a small but significant phase deviation from the classical model. At the $1/e$ field point in the Rayleigh range, there can be as much as a $\lambda/6$ difference between the classical model and the new wave front.

In the course of building this geometrical model, the properties of a skew line are developed in depth; characteristics noted by Arnaud[5] are incorporated, and other characteristics are introduced. In addition, the concept of the skew line as a real ray is explored. Since this idea does have certain drawbacks, the skew line is ultimately used to build a nonorthogonal coordinate system that provides an unambiguous framework for studying Gaussian beam propagation. Finally, it should be pointed out that a ruled surface need not be a figure of revolution and that the skew-line model presented here can be extended to describe elliptic hyperboloids. These are hyperboloids of one sheet whose cross section is an ellipse; they can also be developed from the rotation of a skew line about the z axis. For example, beams with this profile are generated by semiconductor lasers, and a skew-line model can be developed to describe these beams. However, in this paper only those beams with a circular cross section are discussed.

SKEW-LINE GENERATOR OF A RULED SURFACE

In this section, we begin by discussing geometrical characteristics of an individual skew line and its relationship to both the hyperboloid of revolution and the oblate ellipsoid in the oblate spheroidal coordinate system. In subsequent sections we shall expand the discussion to include the behavior of families of skew lines. This constitutes the background for a later section in which the characteristics of a skew line are compared with the ray of geometrical optics.

In Fig. 2, let NP represent a skew line revolving about the z axis, with ON the common perpendicular to the axis and NP in any position. ON then has a constant length W_0, and, as the line NP rotates, the point N describes a circle of radius W_0 in the x–y plane. The angle between the revolving line and the z axis is also a constant δ.

Let $P(x, y, z)$ be any point on the skew line, and let $NP = l$. The parametric equations of the locus of P, in terms of l and ϕ, are then

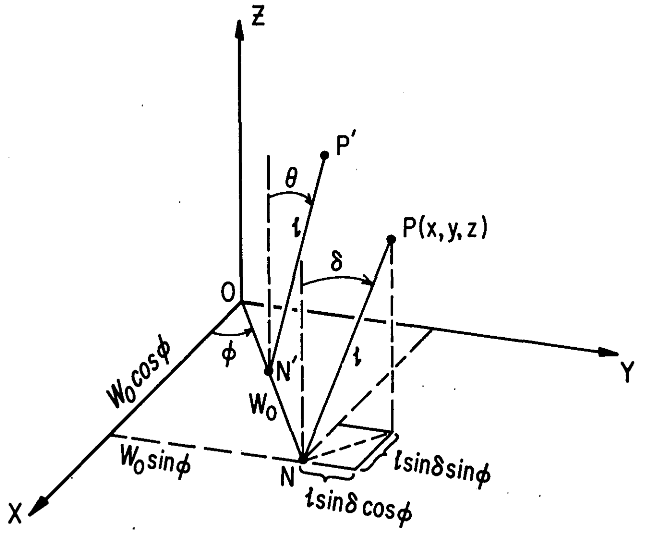

Fig. 2. Orientation of a skew line with respect to the z axis. The skew line, l, is shown for two possible deviation angles, δ and θ.

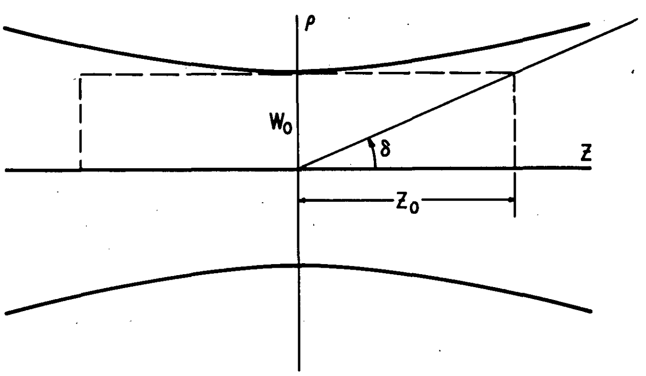

Fig. 3. Cross section of the hyperboloid generated by the rotation of a skew line about the z axis.

$$x = W_0 \cos \phi - l \sin \delta \sin \phi,$$
$$y = W_0 \sin \phi + l \sin \delta \cos \phi,$$
$$z = l \cos \delta. \qquad (7)$$

The point P lies on a surface expressible as an equation in x, y, and z. This equation is independent of l and ϕ, and we can write

$$x^2 + y^2 = W_0{}^2 + l^2 \sin^2 \delta = W_0{}^2 + \frac{z^2 \sin^2 \delta}{\cos^2 \delta},$$

$$x^2 + y^2 - z^2 \tan^2 \delta = W_0{}^2, \qquad (8)$$

$$\frac{x^2 + y^2}{W_0{}^2} - \frac{z^2}{W_0{}^2/\tan^2 \delta} = 1. \qquad (9)$$

This is the equation of a hyperboloid of revolution of one sheet, with a waist radius W_0 and with δ as the angle of the asymptotic cone, or divergence. A cross section of this figure, for ϕ = constant, is shown in Fig. 3.

By the method of Arnaud[5] we can project the skew line NP onto a second plane parallel to the $z = 0$ plane and located a distance $\bar{z}$ away. The projected skew line forms the line segment PQ, as shown in Fig. 4. The angle $PO'Q$ has a constant value α as the skew line rotates about the z axis. As $\bar{z}$ increases, α increases proportionately, while the deviation angle δ remains constant. If $\bar{z}$ remains fixed, various values of α generate different figures of revolution. Figure 5 demonstrates this progression. If $\alpha = 0$, the deviation angle is also zero, and, as the rectilinear generator rotates about the z axis, it sweeps out a cylinder. As α increases, the figure becomes more twisted, generating hyperboloids of revolution of increasing deviation angle and decreasing waist sizes. Finally, in the limit as $\alpha \rightarrow \pi/2$, the waist radius $W_0 \rightarrow 0$, and the figure becomes a right circular cone.

The twist angle α, the deviation angle δ, and the distance from the plane of the waist $\bar{z}$ are related to one another. Referring again to Fig. 4, note that triangle PQO' is a right triangle with line segment PO', given by W, the radius of the beam cross section in the $z = \bar{z}$ plane. Line segments $O'Q$ and PQ then are given by

$$O'Q = W \cos \alpha,$$
$$PQ = W \sin \alpha. \qquad (10)$$

Since $O'Q$ is the projection of the beam-waist radius, W_0, into this second plane, the value $W \cos \alpha$ is always a constant.

Comparing Eq. (9) with Eq. (4), we see that

$$\frac{W_0{}^2}{\tan^2 \delta} = z_0{}^2. \qquad (11)$$

Using Eq. (10), we can rewrite this as

$$\tan \delta = \frac{W \cos \alpha}{z_0}. \qquad (12)$$

The relationship between $\tan \delta$ and $\bar{z}$ can be seen in Fig. 4 and is given by

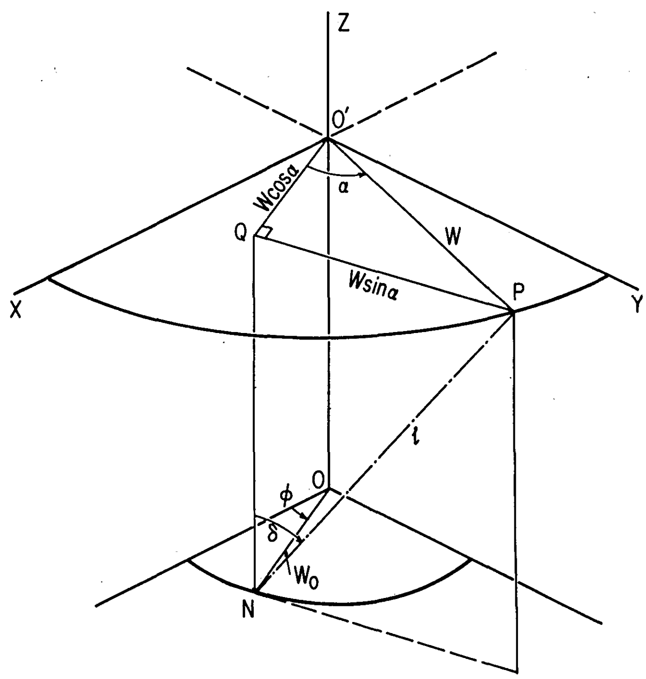

Fig. 4. Skew line projected onto a plane perpendicular to the z axis and a distance NQ from the plane of the waist.

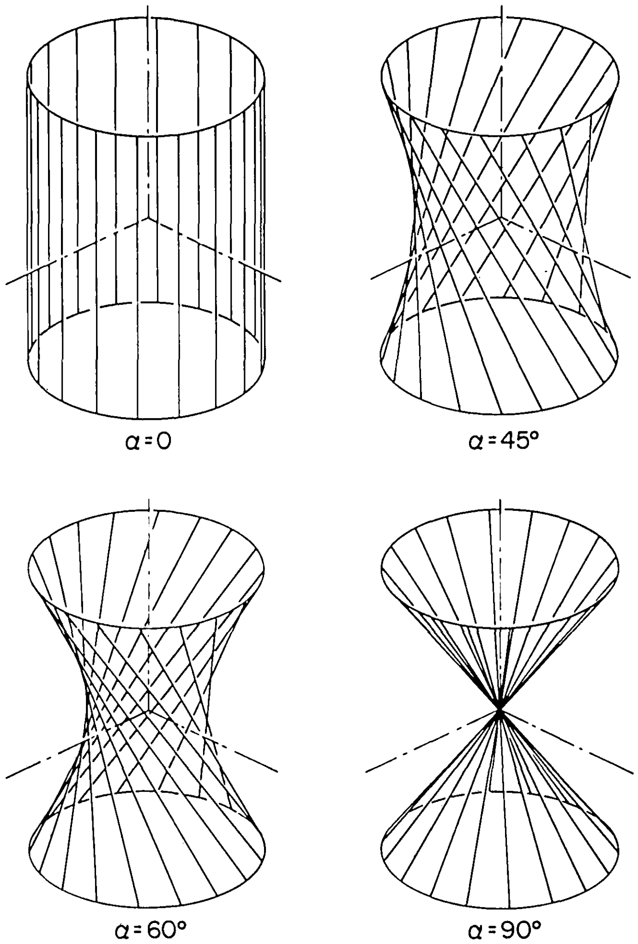

Fig. 5. Different ruled surfaces generated by varying amounts of twist of the rectilinear generator (skew line). For zero twist, the straight line produces a right circular cylinder. As α increases, the figure becomes a hyperboloid of one sheet of increasing divergence until, in the limit of $\alpha = 90°$, it is a right circular cone.

$$\tan \delta = \frac{W \sin \alpha}{\bar{z}}. \tag{13}$$

Now the dependence of the twist angle α on the distance from the plane of the waist can be expressed explicitly by

$$\tan \alpha = \frac{\bar{z}}{z_0}. \tag{14}$$

This is the same result as Eq. (2) with α the same angle as Φ, the phase difference in the classical Gaussian beam description.

As Arnaud pointed out, there are two possible orientations for a skew line; that is, a hyperboloid of revolution can be generated by a skew line deviated to the right of vertical, as shown in Figs. 2 and 4, or by one deviated to the left. This would mean, in the former case, an angle α twisted counterclockwise with respect to the positive z axis and, in the latter case, an angle α twisted clockwise with respect to the z axis. Figure 6 demonstrates this construction. The counterclockwise twist is considered a positive α, since it is the same direction as a positive ϕ. Conversely, the clockwise twist is taken to be a negative α. Although the skew lines are equivalent in the sense that either generates a hyperboloid of one sheet when rotated about the z axis, taken together they generate a set of nonorthogonal parametric lines, or coordinate curves, on the surface of the hyperboloid.

The twist angle, α, can be used to describe the oblate ellipsoid by means of a variety of relationships. We can equate it to the oblate spheroidal coordinate μ, the radius of vertex curvature of the ellipse R_e, and the conic constant K of the ellipse. By doing so, we can relate abstract parameters, such as μ, to a concept that lends itself to physical interpretations. Furthermore, we can develop a basis of comparison with well-known figures, such as circles and spheres, through the relationship of the twist angle to the vertex curvature and the conic constant. Ultimately, this will provide greater insight into the behavior of the Gaussian beam as it propagates. We now examine the relationship between the twist angle and the oblate spheroidal coordinate μ; we shall develop the remaining relationships later in this section.

The oblate spheroidal μ is a hyperbolic angle that can be equated to a circular angle, in this case α, through the gudermanian function.[14] The gudermanian expresses the functional relationship between hyperbolic and circular angles without resorting to imaginary values. This function is written as

$$\alpha = \mathrm{gd}(\mu), \tag{15}$$

where

$$\mathrm{gd}(\mu) = \int_0^{\mu} \frac{\mathrm{d}t}{\cosh(t)}, \tag{16}$$

and the inverse Gudermanian is given by

$$\mu = \mathrm{gd}^{-1}(\alpha) = \int_0^{\alpha} \frac{\mathrm{d}t}{\cos(t)}. \tag{17}$$

Performing the integration in both Eq. (16) and Eq. (17) leads to a pair of inverse relationships

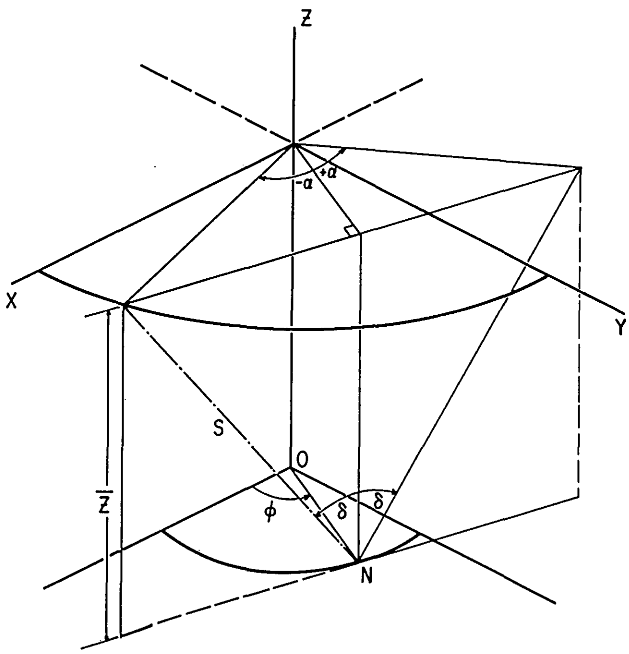

Fig. 6. The two possible twist orientations for the skew line S: a counterclockwise twist for $+\alpha$ and a clockwise twist for $-\alpha$.

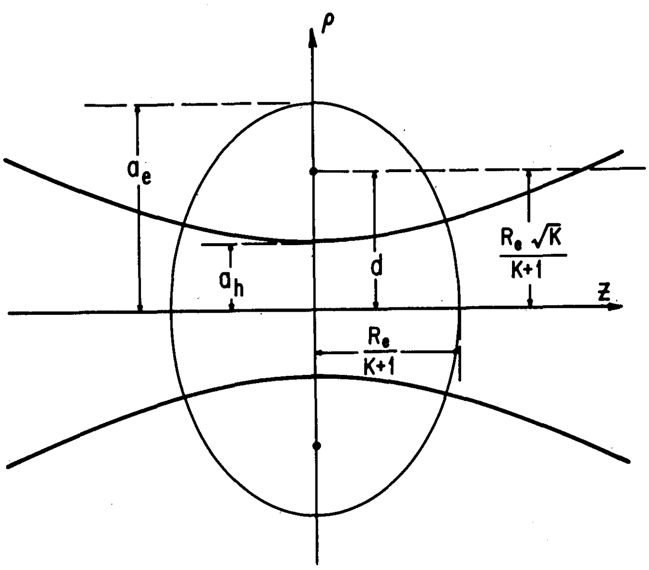

Fig. 7. Cross section of one confocal oblate ellipse and hyperbola. Their common foci are spaced at $2d$; K is the conic constant of the ellipse, and R_e is the radius of the vertex curvature of the ellipse along the semiminor axis.

$$\alpha = 2\tan^{-1}(e^{\mu}) - \pi/2 \tag{18a}$$

and

$$\mu = \ln\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right). \tag{18b}$$

Expanding the tangent function leads to

$$e^{\mu} = \frac{1 + \sin\alpha}{\cos\alpha} \tag{19a}$$

and

$$e^{-\mu} = \frac{1 - \sin\alpha}{\cos\alpha}. \tag{19b}$$

We can now derive a more explicit functional relationship between μ and α. Using the definitions of the oblate spheroidal coordinate ξ given in Appendix A and the exponential definitions of $\sinh\mu$ and $\cosh\mu$, we can write

$$\begin{aligned} \xi &= \sinh\mu = \tan\alpha, \\ (\xi^2 + 1)^{1/2} &= \cosh\mu = 1/\cos\alpha, \\ \tanh\mu &= \sin\alpha. \end{aligned} \tag{20}$$

Using these expressions, we can rewrite the parametric equations for the oblate spheroidal system as

$$\begin{aligned} x &= \frac{d\sin\theta\cos\phi}{\cos\alpha}, \\ y &= \frac{d\sin\theta\sin\phi}{\cos\alpha}, \\ z &= d\tan\alpha\cos\theta. \end{aligned} \tag{21}$$

We can conclude from this analysis that the skew line and its corresponding twist angle can be used in describing the oblate spheroidal coordinate system in which we have described a new mathematical model for a propagating Gaussian beam. Therefore they should be equally useful in describing the beam itself.

In Eq. (9), varying W_0 and δ generates a family of hyperboloids related by their common focus. As is shown in Fig. 2, each member of this family possesses its own rectilinear generator $N'P'$ and deviation angle θ, but all members have common perpendiculars to the z axis, $ON' = W'\cos\alpha$. Let the length of the skew line $N'P' = l$ remain constant for all θ; then the parametric equations of the locus of $P'(x', y', z')$ are similar to Eqs. (7), and we can write

$$\begin{aligned} x' &= W'\cos\phi = W'\cos\alpha\cos\phi - l\sin\theta\sin\phi, \\ y' &= W'\sin\phi = W'\cos\alpha\sin\phi + l\sin\theta\cos\phi, \\ z' &= l\cos\theta. \end{aligned} \tag{22}$$

We wish to find the surface containing P′, which is the endpoint of all skew lines having the same length l and twist α from the plane of the waist. This surface will be independent of θ and ϕ, and we can write

$$\begin{aligned} x'^2 + y'^2 &= W'^2\cos^2\alpha + l^2\sin^2\theta, \\ W'^2 &= W'^2\cos^2\alpha + l^2(1 - z'^2/l^2). \end{aligned} \tag{23}$$

After some algebraic manipulation, Eqs. (23) become

$$\frac{W'^2}{l^2/\sin^2\alpha} + \frac{z^2}{l^2} = 1. \tag{24}$$

This is the equation of an ellipse at a constant distance l, as measured along a skew line, from the plane of the waist of the hyperboloid of revolution. A family of skew lines with constant length l is shown in Fig. 7. The arc PA lies on the ellipse specified by the constant angle α. This family is discussed further in a subsequent section. Since the parameter $l/\sin\alpha > l$ for $\alpha < \pi/2$, this particular ellipse has foci and a semimajor axis located in the plane $z = 0$ and is a figure of revolution about the z axis. As such, it is referred to as an oblate ellipsoid.

Equations (9) and (22) represent specific examples of more-general families of hyperboloids of revolution of one sheet and oblate ellipsoids that are orthogonal. The general equation for a hyperboloid of revolution is given by

$$\frac{\rho^2}{{a_h}^2} + \frac{z^2}{{a_h}^2 - d^2} = 1, \qquad a_h < d, \tag{25}$$

whereas for an ellipse it is

$$\frac{\rho^2}{{a_e}^2} + \frac{z^2}{{a_e}^2 - d^2} = 1, \qquad a_e > d, \tag{26}$$

where $\rho^2 = x^2 + y^2$. For a figure of revolution about z, the focus describes a ring of diameter $2d$. The parameters a_e and a_h, shown in Fig. 8, refer to the semimajor axis of the ellipse and the waist radius of the hyperboloid, respectively.

Comparing Eqs. (22) with Eq. (26), we note that

$${a_e}^2 - d^2 = l^2 = \frac{l^2}{\sin^2\alpha} - d^2. \tag{27}$$

This expression can be reduced to

$$l = d\tan\alpha, \tag{28}$$

and Eq. (26) becomes

$$\frac{\rho^2}{d^2/\cos^2\alpha} + \frac{z^2}{d^2\tan^2\alpha} = 1. \tag{29}$$

Note that the twist angle α determines not only the length of the skew line l but also the specific ellipsoid in a family of

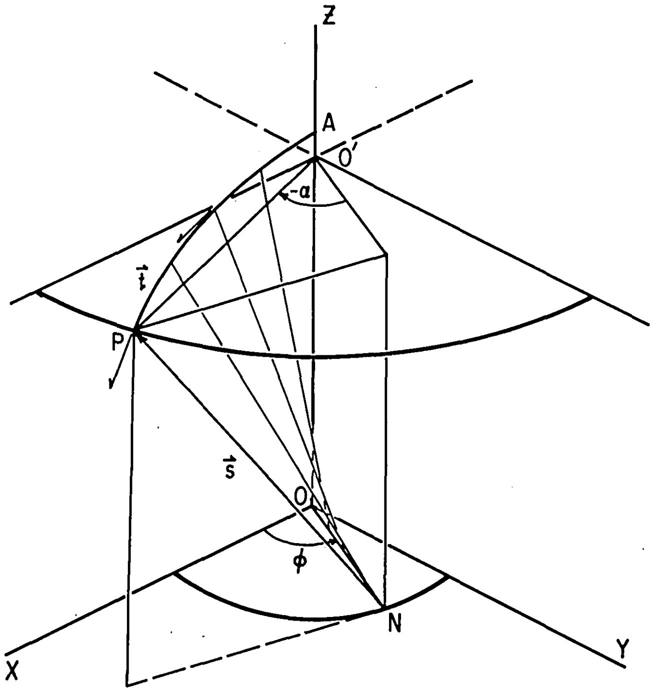

Fig. 8. A fan of skew lines, each having the same length from the elliptical arc PA to the line segment NO. Each skew line is the rectilinear generator for one of the members of the family of hyperboloids in the oblate spheroidal coordinate system. The vector **t** is the tangent vector to the elliptical arc, and **s** is the skew-line vector.

ellipsoids with the same focus. Therefore this angle can be used to ascertain both the conic constant and the radius of vertex curvature associated with each ellipsoidal surface. We now establish the relationship between these two parameters and the twist angle.

Figure 8 demonstrates the connection among the focus distance d, the conic constant K, and the radius of vertex curvature R_e. The conic constant occurs in the expression for the sag of the ellipse and provides a basis of comparison with a sphere, for which $K = 0$. These relationships can be expressed by

$$d^2 = \frac{(R_e{}^2)K}{(K+1)^2} \tag{30}$$

and

$$a_e{}^2 - d^2 = \frac{R_e{}^2}{(K+1)^2}. \tag{31}$$

Comparing Eqs. (26) and (29), we see that the lengths of the skew line and the semiminor axis are equal. Making this substitution into Eq. (31) yields

$$d^2 \tan^2 \alpha = \frac{R_e{}^2}{(K+1)^2}. \tag{32}$$

Using the expression for d^2 given by Eq. (30), we can rewrite Eq.(32) as

$$\tan^2 \alpha = \frac{R_e{}^2}{(K+1)^2} \frac{(K+1)^2}{R_e{}^2 K} = \frac{1}{K}. \tag{33}$$

This leads to expressions for both the sine and the cosine of the twist angle in terms of the conic constant. These expressions are

$$\cos^2 \alpha = \frac{K}{K+1},$$
$$\sin^2 \alpha = \frac{1}{K+1}. \tag{34}$$

A general expression for the hyperboloid of revolution can be obtained by comparing Eqs. (25) and (9). First, note that

$$a_h{}^2 - d^2 = -\frac{w_0{}^2}{\tan^2 \delta} = -\frac{a_h{}^2}{\tan^2 \delta}, \tag{35}$$

where

$$a_h = d \sin \delta. \tag{36}$$

Since δ is a specific deviation angle and θ represents all possible such angles between 0 and $\pm\pi/2$, replacing δ by θ produces a family of hyperboloids expressed by

$$\frac{\rho^2}{d^2 \sin^2 \theta} - \frac{z^2}{d^2 \cos^2 \theta} = 1. \tag{37}$$

In this formulation, the Rayleigh range for any specific hyperboloid is given by

$$z_0 = d \cos \theta, \tag{38}$$

and the expression for the waist radius has the form

$$w_0 = d \sin \theta. \tag{39}$$

So far, we have discussed the characteristics of a skew line as it pertains to both a hyperboloid of one sheet and an oblate ellipsoid. We began by showing how a straight line tilted at an angle δ from the z axis and skewed to it forms a single hyperboloid on rotation about the z axis. Next, we found that the endpoints of a family of skew lines all having the same length and twist formed a single oblate ellipsoid. We then expanded the equations for the hyperboloid to include all possible deviation angles θ between 0 and $\pi/2$ and derived a general equation for a family of hyperboloids with the same focus spacing. We performed the equivalent procedure for a family of ellipsoids. Along the way, we were able to relate the twist angle α of the skew line to the skew-line length, the conic constant of the ellipse, the gudermanian of the hyperbolic angle μ in the oblate spheroidal coordinate system, and the phase difference Φ in the traditional description of the fundamental mode of a Gaussian beam. We can also see from Eqs. (28), (14), and (2) that this phase difference is directly proportional to the skew-line length, or

$$\tan \Phi = \tan \alpha = \frac{l}{d}. \tag{40}$$

We may expand on the relationship between an ellipse, specified by α, and the rectilinear generators of a family of hyperboloids. Each member of this family of skew lines has the same length l and a deviation angle θ appropriate to its hyperboloid, and it intersects the plane of the waist along a line segment at an angle ϕ from the x axis. We shall refer to this family or aggregate of skew lines as a fan, as depicted in Fig. 7. The parametric equations for this family are given by Eqs. (21), with α and ϕ constant. Each skew line, represented here as the vector **s**, has the same length between the elliptical arc PA and the line segment ON. The intersection of each successive skew line with segment ON is a distance $d \sin\theta$ from the origin, where θ is the deviation angle of the new skew line and represents another member in the family of hyperboloids. Furthermore, each skew line is perpendicular to the elliptical arc. This is intuitively true, since each skew line lies on a hyperboloid, the orthogonal surface to an oblate ellipse. A more detailed proof is given in Appendix B.

GEOMETRICAL INTERPRETATION OF $\exp(-i \tan^{-1} \xi/\eta)$

Both the classical mathematical model of a Gaussian beam given in Eq. (1) and the new model of Eq. (6) include arctangent factors in their exponential phase terms. A geometrical interpretation of these terms would provide much useful insight into the nature of a phase front as it propagates. In the case of the term $\exp(-i \tan^{-1} \xi/\eta)$, a simple geometrical explanation does exist.

Figure 9 displays the skew line PN once again at a deviation angle θ, along with the attendant elliptical arc PA. Further, the figure designates two planes, one perpendicular to the z axis at the point O′ and the other perpendicular to the z axis at point A. The O′ plane intersects the hyperbola of deviation angle θ in a circle with radius $\rho = PO'$. The A plane is the tangent plane to this oblate ellipsoid at the point (O, O, A). The distance along the z axis between these two planes is the sag of ellipse. As demonstrated earlier in this paper, the distance AO must be the length of the skew line, $d \tan\alpha$. Therefore the equation for the sag is given by

$$\text{sag} = d \tan\alpha(1 - \cos\theta). \tag{41}$$

The distance from the plane O′ to plane A along the skew line, designated Δl in the drawing, is simply sag/cos θ, or

$$\Delta l = \frac{d \tan\alpha}{\cos\theta}(1 - \cos\theta). \tag{42}$$

The increased length of the skew line translates directly into an increase in the twist angle α. We shall refer to this increased twist as $\Delta\alpha$. The entire length of the skew line from the point N until it intersects the plane A is given by $l + \Delta l$. Given in terms of the twist angle, this is

$$l + \Delta l = d \tan(\alpha + \Delta\alpha). \tag{43}$$

Substituting the value of $d \tan\alpha$ for l and Eq. (42) for Δl yields

$$d \tan\alpha + \frac{d \tan\alpha}{\cos\theta} - d \tan\alpha = d \tan(\alpha + \Delta\alpha). \tag{44}$$

This reduces to

$$\frac{\tan\alpha}{\cos\theta} = \tan(\alpha + \Delta\alpha). \tag{45}$$

From the definition of the oblate spheroidal coordinates, we can substitute ξ and η for the terms $\tan\alpha$ and $\cos\theta$, respectively, and Eq. (45) becomes

$$\frac{\xi}{\eta} = \tan(\alpha + \Delta\alpha). \tag{46}$$

Finally, by taking the arctangent of both sides, we obtain

$$\tan^{-1} \xi/\eta = \alpha + \Delta\alpha. \tag{47}$$

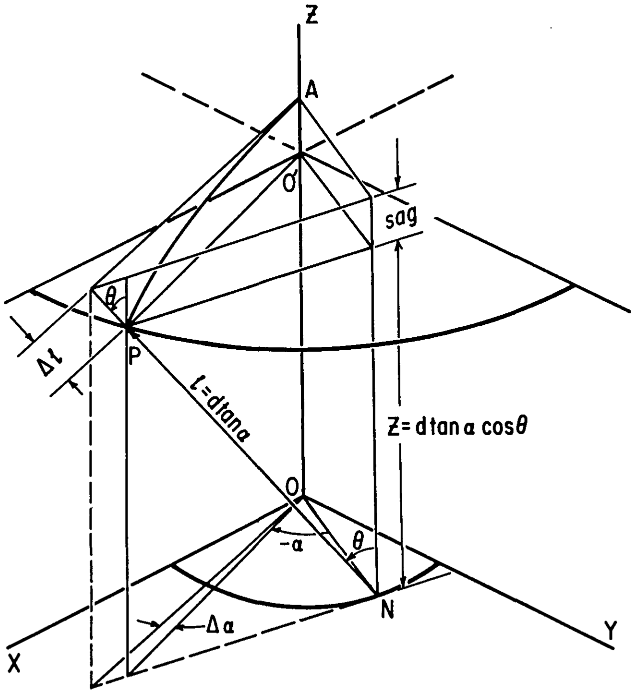

Fig. 9. The differential change in twist angle, $\Delta\alpha$, corresponds to the sag of the elliptical arc, PA, as measured along the skew line. This distance is given by $d\xi/\eta - d\xi$.

Some algebraic manipulation is required to obtain an expression for $\Delta\alpha$ in terms of ξ and η. This equation is

$$\Delta\alpha = \tan^{-1}\left[\frac{\xi(1-\eta)}{\eta + \xi^2}\right]. \tag{48}$$

The entire exponential phase factor $\exp(-i \tan^{-1} \xi/\eta)$ can then be rewritten as

$$\exp(-i \tan^{-1} \xi/\eta) = \exp\left[-i \tan^{-1} \xi - \tan^{-1} \frac{\xi(1-\eta)}{\eta + \xi^2}\right]. \tag{49}$$

The difference between the exponential arctangent phase term of the classical Gaussian beam mathematical model and the one represented by Eq. (6) is the phase term attributable to $\Delta\alpha$.

The exponential factor $\exp(-i \tan^{-1} \xi/\eta)$ can be interpreted as representing the difference in phase that a wave disturbance would undergo in traveling along the skew line in the time required for the edge of the beam to advance to the z distance represented by plane A. In other words, as the beam propagates, its center passes plane A at time t_1. At a later time t_2, the beam edge will pass plane A, and in the time $t_2 - t_1$, the phase of the beam will have changed by $\Delta\alpha$. Another interpretation is that of an off-axis, or wave-front, error; that is, a wave front is ideally a section of an oblate ellipse, but off axis it deviates from the ideal by an amount given by $\Delta\alpha/k$. For example, $\alpha = 45°$ at the Rayleigh range of any Gaussian beam. In the case of a highly divergent beam with a θ of 10° at the $1/e$ field point, the wave front has a $\lambda/8$ deviation from a perfectly oblate ellipse at the $1/e$ radius at the Rayleigh range. In the limit as $\eta \to 0$, the skew line collapses to the plane of the waist, and the exponential factor equals $\exp(-i\pi/2)$. Since the skew line normally extends on either side of the waist, and since the entire line collapses into the beam waist as $\eta \to 0$, the overall phase change is 180°. This is consistent with the 180° phase shift experienced by a general light wave in passing through a focus. Finally, we note that the sign of the phase term is such that a surface of constant phase will be curved toward the plane of the waist and inside the figure of the oblate ellipse. Although it is tempting to believe that this makes the wave front more nearly spherical, the opposite is true. An oblate ellipse is always curved inside the figure of a sphere whose radius corresponds to the vertex curvature of the ellipse.

THE SKEW LINE AS A RAY

We have pointed out a number of interesting features of individual skew lines as well as of fans of skew lines. In particular, we note that the relationship between an elliptical arc and a fan of skew lines demonstrates three characteristics of wave fronts and their trajectories. First, the spatial separation as measured along any skew line of the fan between an elliptical arc and the plane of the waist remains constant. This statement can be generalized to include the separation between any two elliptical arcs that are different twist angles and are perpendicular to the same family of skew lines. Second, successive elliptical arcs are perpendicular to a skew line, prompting comparisons between successive arcs on a wave front and the wave front's orthogonal trajectory, or ray. Finally, the skew line is a straight line, as is the current geometrical-optics model ray in a homogeneous medium.

Still further, we can prove that, when a ray with a skew-line trajectory is reflected from an oblate ellipsoidal mirror, the reflected ray also possesses a skew-line trajectory on the same hyperboloid as the incident ray. Proof of this requires a three-dimensional vector analysis and can be found in Appendix C. This particular characteristic of the skew line finds a useful application in modeling ray behavior in an optical resonator. The continuum of all such skew rays forms the hyperbolic amplitude contours. Such an envelope of rays was discussed by Bykov and Vainshtein,[15] Kahn,[16] and Stein.[17]

All the above results demonstrate that the skew-line model of a propagating Gaussian beam, as expressed in Eq. (6), possesses strong parallels with the theorems of geometrical optics. However, two traits of a skew line prevent it from being defined as a ray. The first is that the skew line is not the gradient of the wave described by Eq. (6). Second, if a skew line is to be seen as a ray, the possibilities of its two different orientations must be considered equally likely. One way to include both positive and negative (counterclockwise and clockwise) twists in the geometrical description of a Gaussian beam is to treat the two skew lines as unit coordinate vectors, $\hat{u}$ and $\hat{v}$, which, together with the unit vector $\hat{\theta}$, form a nonorthogonal coordinate system.

The skew-line fan of Fig. 7 can be repeated for a counterclockwise twist, and the resultant figure forms a mirror image of the clockwise fan. This is demonstrated in Fig. 10. The skew line **s** of Fig. 7 reappears as $\hat{v}$ pointed in the opposite direction, and the tangent vector to the ellipse, **t**, has been renamed $\hat{\theta}$. The skew line for a positive α, or a counterclockwise twist, is $\hat{u}$. Figure 10 sketches the skew-line fans for a few discrete deviation angles. In actual prac-

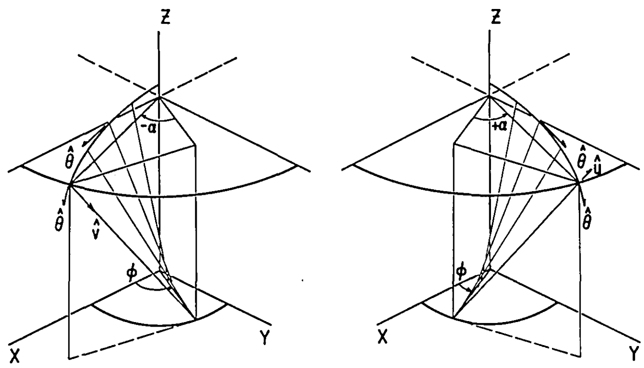

Fig. 10. Skew-line fans for a clockwise twist ($-\alpha$) and a counterclockwise twist ($+\alpha$), for discrete skew lines. The skew lines are denoted as the vectors $\hat{v}$ and $\hat{u}$, and $\hat{\theta}$ is the tangential vector to the elliptical arc. The figures are mirror images of each other.

tice, the fans are surfaces called right conoids, as shown in Fig. 11. All these conoids are ruled surfaces; they are formed by the motion of a straight line, the skew line, in three-dimensional space. Together with the oblate ellipsoid, these surfaces form three coordinate surfaces in a nonorthogonal coordinate system, which we now explore. An excellent discussion of general three-dimensional curvilinear coordinate systems can be found in Stratton's *Electromagnetic Theory*.[18]

NONORTHOGONAL COORDINATE SYSTEM

The coordinate system whose unit base vectors are shown in Fig. 12 consists of two right conoids, one left-handed and the other right-handed, whose axis of symmetry is the z axis, and an oblate elliposid with a focal ring of radius d in the x–y plane. The right-handed conoid is the counterclockwise skew-line fan ($+\alpha$), so named because the skew lines seem to sweep in the direction of the fingers of the right hand with the thumb pointed in the direction of the positive z axis as α varies from $-\pi/2$ to $+\pi/2$. Similarly, the left-handed conoid is the clockwise skew-line fan ($-\alpha$). This time, as α varies from $+\pi/2$ to $-\pi/2$, the skew lines sweep in the direction of the fingers of the left hand when the thumb is pointed in the direction of the positive z axis. The coordinate angle θ is the deviation angle of the skew line with respect to the z axis. This coordinate system is not orthogonal, since the coordinate surfaces do not intersect at right angles.

The parametric equations for this system are

$$x(u, v, \theta) = \frac{d \sin\theta \cos\left[\dfrac{u+v}{2}\right]}{\cos\left[\dfrac{u-v}{2}\right]},$$

$$y(u, v, \theta) = \frac{d \sin\theta \sin\left[\dfrac{u+v}{2}\right]}{\cos\left[\dfrac{u-v}{2}\right]},$$

$$z(u, v, \theta) = d \cos\theta \tan\left[\frac{u-v}{2}\right], \tag{50}$$

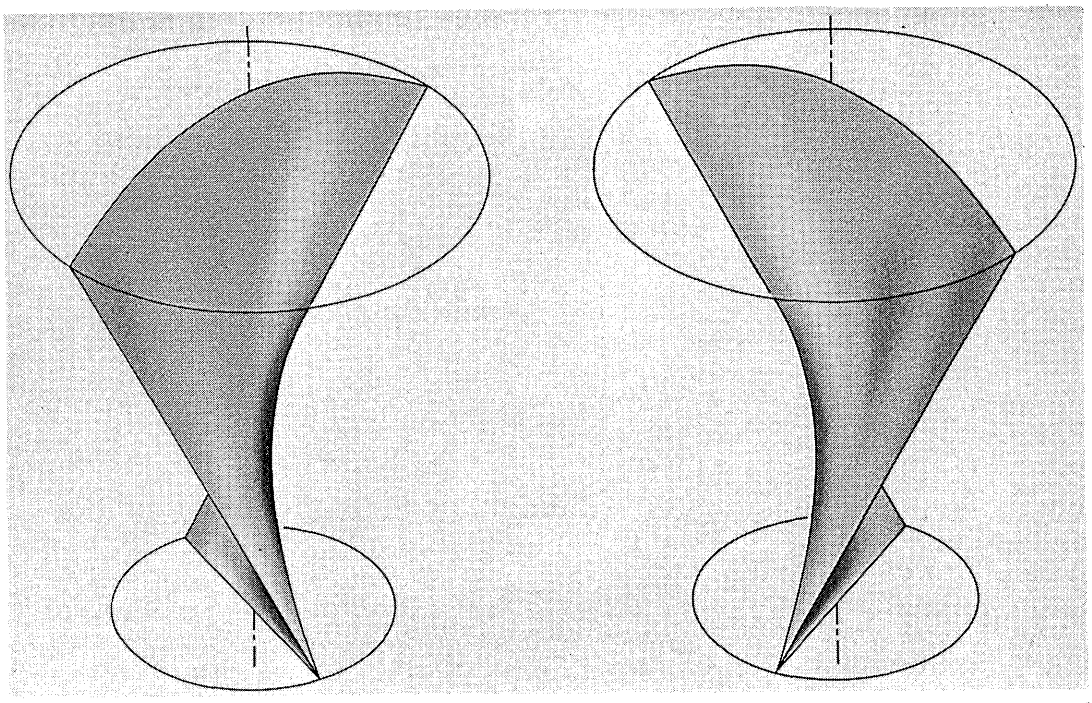

Fig. 11. Continuous surfaces generated by the skew-line fans of Fig. 10. The left-hand drawing represents a left-handed right conoid corresponding to a clockwise twist, and the right-hand drawing represents a right-handed right conoid corresponding to a counterclockwise twist. In both drawings, the lower circle represents the plane of the waist, and the upper circle represents an oblate ellipsoid a distance from the waist. The vertical dashed–dotted line represents the z axis in both drawings.

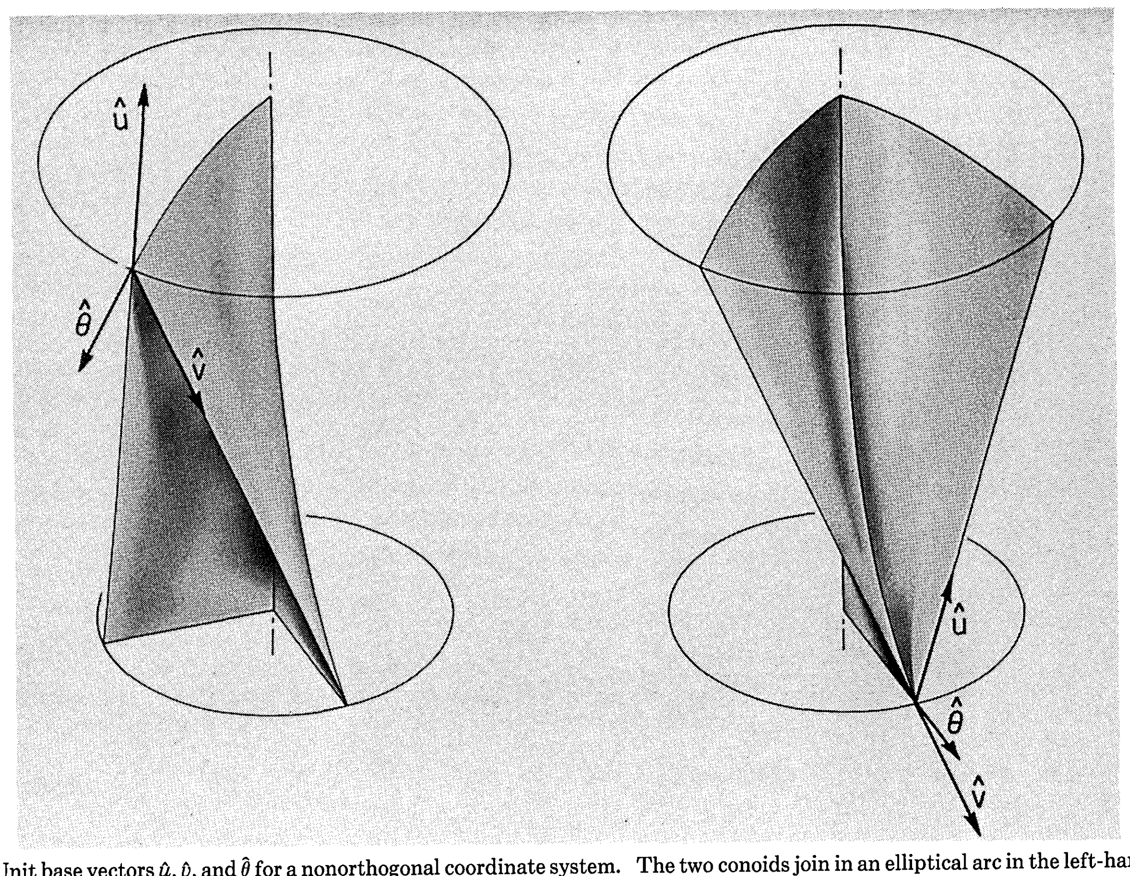

Fig. 12. Unit base vectors $\hat{u}$, $\hat{v}$, and $\hat{\theta}$ for a nonorthogonal coordinate system. The two conoids join in an elliptical arc in the left-hand drawing. Their junction in the plane of the waist is shown in the right-hand drawing.

where

$$u = \phi + \alpha \tag{51}$$

and

$$v = \phi - \alpha. \tag{52}$$

Note that these equations strongly resemble the parametric equations of the oblate spheroidal coordinate system. For that system, we discussed surfaces on which one of the coordinate variables was constant and the unit base vectors, $\hat{\xi}$, $\hat{\eta}$, and $\hat{\phi}$, were the normals to these surfaces. For the sake of simplicity, we shall not discuss the surfaces on which the coordinates u, v, and θ are constant. Instead we shall outline the unit base vectors, u, v, and θ, as the vectors tangent to the right-handed conoid, the left-handed conoid, and the oblate ellipsoid, respectively.

The position vector $\mathbf{r}$ to any point in the three-dimensional space is given by

$$\mathbf{r} = x(u, v, \theta)\hat{x} + y(u, v, \theta)\hat{y} + z(u, v, \theta)\hat{z}, \tag{53}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit base vectors in the Cartesian coordinate system. A differential change in $\mathbf{r}$, due to small displacements along the coordinate curves, is expressed by

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\,\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}v + \frac{\partial \mathbf{r}}{\partial \theta}\,\mathrm{d}\theta. \tag{54}$$

If one moves a unit distance along any one of the coordinate curves, the change in $\mathbf{r}$ is directed tangentially along that curve and has a magnitude of 1. The vectors

$$\hat{u} = \frac{\dfrac{\partial \mathbf{r}}{\partial u}}{\left|\dfrac{\partial \mathbf{r}}{\partial u}\right|},$$

$$\hat{v} = \frac{\dfrac{\partial \mathbf{r}}{\partial v}}{\left|\dfrac{\partial \mathbf{r}}{\partial v}\right|},$$

$$\hat{\theta} = \frac{\dfrac{\partial \mathbf{r}}{\partial \theta}}{\left|\dfrac{\partial \mathbf{r}}{\partial \theta}\right|} \tag{55}$$

are the unit base vectors for the coordinate system. For the nonorthogonal system here, these are

$$\hat{u} = (-\sin\theta \sin v)\hat{x} + (\sin\theta \cos v)\hat{y} + (\cos\theta)\hat{z},$$
$$\hat{v} = (-\sin\theta \sin u)\hat{x} + (\sin\theta \cos u)\hat{y} + (\cos\theta)\hat{z},$$
$$\hat{\theta} = \frac{\cos\theta \cos\phi}{(1 - a^2)^{1/2}}\,\hat{x} + \frac{\cos\theta \sin\phi}{(1 - a^2)^{1/2}}\,\hat{y} - \frac{\sin\theta \sin[(u - v)/2]}{(1 - a^2)^{1/2}}\,\hat{z}, \tag{56}$$

where $a = \sin\theta \cos[(u - v)/2]$. Figure 12 shows these vectors at two different points in space.

In the left-hand drawing of Fig. 12, the observation point resides on an oblate ellipse some distance from the plane of the waist. The vectors $\hat{u}$, $\hat{v}$, and $\hat{\theta}$ are shown as tangents to the two conoids and the oblate ellipsoid, respectively. The conoidal surfaces meet in an arc on this ellipsoid and are separated in the plane of the waist by the acute angle 2α, where α is the twist angle associated with the ellipsoid. Likewise, the right-hand drawing in Fig. 12 shows the observation point in the plane of the waist with the associated unit base vectors. The conoids meet along a line segment in the waist and are separated on an oblate ellipsoid by the acute angle 2α.

The most useful aspect of these unit vectors is their relationship to the unit vectors of the oblate spheroidal system. Starting with Eqs. (56), some algebraic manipulation leads to the relations

$$\hat{\xi} = \frac{\hat{u} - \hat{v}}{2(1 - a^2)^{1/2}},$$
$$\hat{\phi} = \frac{\hat{u} + \hat{v}}{2a},$$
$$\hat{\eta} = -\hat{\theta}. \tag{57}$$

Clearly, the connection between the two systems is a simple one, with the useful result that two straight-line vectors, $\hat{u}$ and $\hat{v}$, can synthesize the tangential vectors, $\hat{\xi}$ and $\hat{\phi}$, to two curves in space, a hyperbola and a circle.

Although using a nonorthogonal coordinate system seems an unnecessary complication of the geometrical model, this particular coordinate system has some advantages over the oblate spheroidal coordinate system in describing a propagating Gaussian beam. Specifically, a wave-front surface normal $\hat{\xi}$ can be obtained from the difference between the two skew-line vectors, $\hat{u}$ and $\hat{v}$. This gives us the advantage of using straight-line trajectories to predict the beam's behavior on propagation without neglecting the proper description of a wave-front normal. Also, the nonorthogonal system is allied so closely with the oblate spheroidal coordinate system that straight-line propagation can be used without deviating from an exact mathematical description of the beam.

CONCLUSION

A simple but precise geometrical interpretation of the alternative mathematical description of the zero-order mode of a propagating Gaussian beam, represented by Eq. (6), is presented in this paper. Both the geometrical and the mathematical models are expressed in the oblate spheroidal coordinate system, and it is one of the coordinate surfaces, the hyperboloid of one sheet, of this system that forms the basis of the geometrical configuration. Specifically, the hyperboloid is an example of a ruled surface that can be generated by a straight line, skewed to the z axis, rotating about the z axis.

The properties of an individual skew line as well as those of families of skew lines have been discussed in detail. We began by describing the hyperboloid generated by the skew line and then related the skew line to the oblate ellipsoid with the twist angle concept. This relationship exists in the definition of the conic constant of the ellipse, the gudermanian of the hyperbolic angle μ in the oblate spheroidal coordinate system, and in the length of the semiminor axis of the ellipse. Furthermore, the length of the skew line from the plane of the waist to an ellipsoidal surface is directly proportional to the tangent of the twist angle.

One intriguing aspect of both the new and the traditional models of the Gaussian beam is the presence of a pure phase term that has an arctangent dependence. In the traditional model, the phase term is $\exp(-i\alpha)$, where α is the twist angle described in an earlier section. The interpretation of this term is a phase difference between the actual wave front and a plane wave. In Eq. (6), however, the term appears as $\exp(-i \tan^{-1} \xi/\eta)$, which represents the sag of the oblate ellipse as measured along the skew line.

The properties of an individual skew line can be expanded to include families of these lines, and it is in consideration of one type of family in particular, the fan, that strong parallels with the ray of geometrical optics can be found. A fan of skew lines intersects any oblate ellipse in an arc, and each member of the fan is perpendicular to this arc. Further, the spatial separation remains constant between two successive arcs, as measured along any member of the fan, much like the constant separation between any two successive wave fronts, as measured along their orthogonal trajectories or rays. Proceeding with the idea of a skew line as a ray, it is shown in Appendix C that a ray with skew-line trajectory reflected from a mirror that is a section of an oblate ellipsoid yields its opposite member, that is, a ray with the skew-line trajectory of the same hyperboloid and an equal, but opposite, twist angle.

Finally, we pointed out the failure of the skew line as a ray. Specifically, the skew line is not the gradient of the wave described by Eq. (6), and since there are two possible orientations of each skew line, they must be considered equally likely. It was then suggested that the real power of the skew-line model might be shown by using both orientations as two components in a nonorthogonal system, and the details of this system were discussed. Instead of creating an unnecessary complication of the issue, the use of both of these skew-line vectors makes straight-line propagation possible, while providing the unambiguous framework necessary for locating points in space, identifying the wave-front normal, and defining deformed surfaces such as aberrated wave fronts.

APPENDIX A: THE OBLATE SPHEROIDAL COORDINATE SYSTEM

The oblate spheroidal coordinate system, shown in Fig. 1, is formed by rotating a system of mutually orthogonal ellipses and hyperbolas about the minor axis of the ellipse. The z axis is the axis of rotation, and the focus is a ring of radius d in the x–y plane. The parametric equations relating the oblate spheroidal coordinate system to Cartesian coordinates are

$$\begin{aligned} x &= d \cosh\mu \sin\theta \cos\phi, \\ y &= d \cosh\mu \sin\theta \sin\phi, \\ z &= d \sinh\mu \cos\theta, \end{aligned} \tag{A1}$$

with either

$$0 \le \theta \le \pi, \quad 0 \le \mu < \infty, \quad 0 \le \phi \le 2\pi \tag{A2a}$$

or

$$0 \le \theta \le \pi/2, \quad -\infty \le \mu < \infty, \quad 0 \le \phi \le 2\pi. \tag{A2b}$$

In the oblate case, we let $\xi = \sinh\mu$ and $\eta = \cos\theta$. The parametric equations then become

$$\begin{aligned} x &= d(1+\xi^2)^{1/2}(1-\eta^2)^{1/2}\cos\phi, \\ y &= d(1+\xi^2)^{1/2}(1-\eta^2)^{1/2}\sin\phi, \\ z &= d\xi\eta, \end{aligned} \tag{A3}$$

with either

$$-1 \le \eta \le 1, \quad 0 \le \xi < \infty, \quad 0 \le \phi \le 2\pi \tag{A4a}$$

or

$$-0 \le \eta \le 1, \quad -\infty \le \xi < \infty, \quad 0 \le \phi \le 2\pi. \tag{A4b}$$

In the oblate system, the surface $|\xi|$ = constant > 0 is an oblate ellipsoid with a major axis of length $2d \cosh\mu$ and minor axis of length $2d|\sinh\mu|$. The surface $\xi = 0$ is a circular disk of radius d centered at the origin in the x–y plane. The surface $|\eta|$ = constant < 1 is a hyperboloid of revolution of one sheet whose asymptotes pass through the origin, inclined at an angle $\theta = \cos^{-1}\eta$ to the z axis. The degenerate surface $\eta = 1$ is the z axis. The surface $\eta = 0$ is the x–y plane, except for the circular disk $\xi = 0$. Finally, the surface ϕ = constant is the azimuthal plane containing the z axis. The angle ϕ is measured from the x–z plane.

APPENDIX B: PROOF THAT AN ARC OF AN OBLATE ELLIPSE IS PERPENDICULAR TO ITS ATTENDANT SKEW-LINE FAN

The parametric Eqs. (A1) describe the arc PA in Fig. 8 for the case in which α and ϕ are held constant. These equations are therefore functions of θ only and have the form

$$\begin{aligned} x(\theta) &= \frac{d \sin\theta}{\cos\alpha}\cos(\phi \pm \alpha), \\ y(\theta) &= d\frac{\sin\theta}{\cos\alpha}\sin(\phi \pm \alpha), \\ z(\theta) &= d \tan\alpha \cos\theta, \end{aligned} \tag{B1}$$

where we have made the substitutions for $\sinh\mu$ and $\cosh\mu$ given in Eqs. (20).

The angle ϕ is measured from the x axis to the line segment ON in the plane of the waist only. The azimuthal angle in any other plane perpendicular to the z axis is given by $\phi \pm \alpha$. The angle α determines the ellipse of interest and the length of the skew line. The tangent to the elliptical arc is given by

$$\mathbf{t} = \frac{\partial \mathbf{r}}{\partial\theta} = \frac{\partial x(\theta)}{\partial\theta}\hat{x} + \frac{\partial y(\theta)}{\partial\theta}\hat{y} + \frac{\partial z(\theta)}{\partial\theta}\hat{z}, \tag{B2}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are Cartesian unit vectors. Differentiating with respect to θ yields

$$\mathbf{t} = \frac{d \cos\theta \cos(\phi \pm \alpha)}{\cos\alpha}\hat{x} + \frac{d\cos\theta \sin(\phi \pm \alpha)}{\cos\alpha}\hat{y} - d\tan\alpha \sin\theta\,\hat{z}. \tag{B3}$$

The $\pm$ for α determines whether the arc in question is for a counterclockwise ($+\alpha$) twist or a clockwise ($-\alpha$) one.

The vector representation for a skew line $\mathbf{s}$ is given by

$$\mathbf{s} = (x' - x)\hat{x} + (y' - y)\hat{y} + (z' - z)\hat{z}. \tag{B4}$$

In the plane of the waist, $\alpha = 0$, and the parametric Eqs. (B1) become

$$x = d \sin\theta \cos\phi,$$
$$y = d \sin\theta \sin\phi,$$
$$z = 0. \quad \text{(B5)}$$

On the elliptical arc, the coordinates of a point are x', y', and z' and are described by the parametric equations

$$x'(\theta) = \frac{d \sin\theta \cos(\phi \pm \alpha)}{\cos\alpha},$$
$$y'(\theta) = \frac{d \sin\theta \sin(\phi \pm \alpha)}{\cos\alpha},$$
$$z'(\theta) = d \tan\alpha \cos\theta. \quad \text{(B6)}$$

Substituting Eqs. (B5) and (B6) into Eq. (B4) gives

$$\mathbf{s} = (\mp d \tan\alpha \sin\theta \sin\phi)\hat{x} \pm (d \tan\alpha \sin\theta \cos\phi)\hat{y} + (d \tan\alpha \cos\theta)\hat{z}. \quad \text{(B7)}$$

The magnitude of $\mathbf{s}$ is $d \tan\alpha$, the length of the skew line. Note that, regardless of the choice of α, the magnitudes of the $\hat{x}$ and $\hat{y}$ components are opposite in sign.

Taking the dot product of $\mathbf{t}$ and $\mathbf{s}$ leads to

$$\mathbf{t}\cdot\mathbf{s} = \mp d^2 \tan\alpha \frac{\sin\theta\cos\theta}{\cos\alpha}\cos(\phi \pm \alpha)\sin\phi \pm d^2 \tan\alpha \frac{\sin\theta\cos\theta}{\cos\alpha}\sin(\phi \pm \alpha) \times \cos\phi - d^2 \tan^2\alpha \sin\theta\cos\theta. \quad \text{(B8)}$$

Simplifying terms results in

$$\mathbf{t}\cdot\mathbf{s} = d^2 \frac{\tan\alpha}{\cos\alpha}\sin\theta\cos\theta\,[\mp\sin\phi\cos(\phi \pm \alpha) \pm \cos\phi\sin(\phi \pm \alpha) - \sin\alpha]. \quad \text{(B9)}$$

Next, we expand the term in brackets to find that

$$\mathbf{t}\cdot\mathbf{s} = d^2 \frac{\tan\alpha}{\cos\alpha}\sin\theta\cos\theta\,[\pm\sin\phi(\cos\phi\cos\alpha \mp \sin\phi\sin\alpha) \pm \cos\phi(\sin\phi\cos\alpha \pm \cos\phi\sin\alpha) - \sin\alpha]$$
$$= \frac{d^2\tan\alpha}{\cos\alpha}\sin\theta\cos\theta[\sin\alpha(\mp\sin^2\phi \mp \cos^2\phi) - \sin\alpha], \quad \text{(B10)}$$

and, finally,

$$\mathbf{t}\cdot\mathbf{s} = 0. \quad \text{(B11)}$$

Therefore the skew line is perpendicular to the elliptical arc *PA*.

APPENDIX C: REFLECTION OF A SKEW-LINE RAY FROM AN ELLIPTICAL MIRROR

In geometrical optics, the laws for reflection are

1. The reflected ray lies in the plane formed by the incident ray and the surface normal.
2. The reflected ray forms an angle to the normal that is equal but opposite that of the incident ray to the normal.

Since the reflective surface is an oblate ellipse, its normal is simply the unit normal for the oblate ellipsoid $\hat{\xi}$ in the oblate spheroidal coordinate system. The three unit normals for this system are given by

$$\hat{\xi} = \frac{\xi(1-\eta^2)^{1/2}}{(\xi^2+\eta^2)^{1/2}}\cos(\phi \pm \alpha)\hat{x} + \frac{\xi(1-\eta^2)^{1/2}}{(\xi^2+\eta^2)^{1/2}}\sin(\phi \pm \alpha)\hat{y} + \eta\frac{(\xi^2+1)^{1/2}}{(\xi^2+\eta^2)^{1/2}}\hat{z},$$
$$\hat{\eta} = -\eta\frac{(\xi^2+1)^{1/2}}{(\xi^2+\eta^2)^{1/2}}\cos(\phi \pm \alpha)\hat{x} - \eta\frac{(\xi^2+1)^{1/2}}{(\xi^2+\eta^2)^{1/2}}\sin(\phi \pm \alpha)\hat{y} + \xi\frac{(1-\eta^2)^{1/2}}{(\xi^2+\eta^2)^{1/2}}\hat{z},$$
$$\hat{\phi} = -\sin(\phi \pm \alpha)\hat{x} + \cos(\phi \pm \alpha)\hat{y}, \quad \text{(C1)}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit vectors in the Cartesian coordinate system. We have made use here of the definition of the azimuthal angle $\phi \pm \alpha$ in a plane other than the plane of the waist. Furthermore, we shall use the definitions for $\tan\alpha$ and $\cos\alpha$ given in Eq. (20).

Since the incident ray has the same trajectory as a skew line, its vector representation can be found from Eq. (B7). For a skew-line trajectory with a clockwise twist ($-\alpha$), the incident ray is the unit vector

$$\hat{u} = (\sin\theta\sin\phi)\hat{x} - (\sin\theta\cos\phi)\hat{y} + (\cos\theta)\hat{z}. \quad \text{(C2)}$$

In order to find the plane containing $\hat{\xi}$ and $\hat{u}$, it suffices to find the normal to both $\hat{\xi}$ and $\hat{u}$, since this is also the normal to the plane containing the two vectors. The unit normal can be found by taking the cross product of $\hat{\xi}$ for a negative α and $\hat{u}$, thereby obtaining

$$\frac{\hat{u}\times\hat{\xi}}{|\hat{u}\times\hat{\xi}|} = -\frac{\eta\cos(\phi-\alpha)(\xi^2+1)^{1/2}}{(\xi^2+\eta^2)^{1/2}}\hat{x} - \frac{\eta\sin(\phi-\alpha)(\xi^2+1)^{1/2}}{(\xi^2+\eta^2)^{1/2}}\hat{y} + \frac{\xi(1-\eta^2)^{1/2}}{(\xi^2+\eta^2)^{1/2}}\hat{z}. \quad \text{(C3)}$$

This is simply $\hat{\eta}$.

Next, we calculate the angle between the incident ray and the surface normal, which is

$$\cos^{-1}(\hat{u}\cdot\hat{\xi}) = \cos^{-1}\left[\frac{\sin^2\alpha\sin^2\theta + \cos^2\theta}{\cos\alpha(\tan^2\alpha + \cos^2\theta)^{1/2}}\right]. \quad \text{(C4)}$$

Reduction of the term on the right-hand side leads to

$$\cos^{-1}(\hat{u}\cdot\hat{\xi}) = \cos^{-1}[(\sin^2\alpha + \cos^2\theta\cos^2\alpha)^{1/2}]. \quad \text{(C5)}$$

In determining the reflected ray $\hat{v}$, we know that it must lie in the plane formed by $\hat{u}$ and $\hat{\xi}$ and must therefore be perpendicular to $\hat{\eta}$. Also, the angle that $\hat{v}$ makes with $\hat{\xi}$ must be equal to and opposite $\cos^{-1}(\hat{u}\cdot\hat{\xi})$. Expressed in terms of a dot product, the latter condition is

$$\hat{v}\cdot\hat{\xi} = -\hat{u}\cdot\hat{\xi}. \quad \text{(C6)}$$

Finally, the angle formed by $\hat{u}$ and $\hat{v}$ must be twice that of $\hat{u}\cdot\hat{\xi}$. This condition can also be expressed in terms of a dot product as

$$\hat{u}\cdot\hat{v} = 1 - 2(\hat{u}\cdot\hat{\xi})^2. \quad \text{(C7)}$$

Taken together, these conditions will produce the three direction cosines of the vector $\hat{v}$.

If the vector $\hat{v}$ is given by

$$\hat{v} = a\hat{x} + b\hat{y} + c\hat{z}, \tag{C8}$$

then the three simultaneous equations to find a, b, and c are

$$\begin{aligned}
&-\cos\theta\cos(\phi-\alpha)a - \cos\theta\sin(\phi-\alpha)b \\
&\qquad + (\sin\alpha\sin\theta)c = 0, \\
&\sin\alpha\sin\theta\cos(\phi-\alpha)a + \sin\alpha\sin\theta\sin(\phi-\alpha)b \\
&\qquad + (\cos\theta)c = -\sin^2\alpha - \cos^2\theta\cos^2\alpha, \\
&(\sin\theta\sin\phi)a - (\sin\theta\cos\phi)b \\
&\qquad + (\cos\theta)c = 1 - 2(\sin^2\alpha + \cos^2\theta\cos^2\alpha).
\end{aligned} \tag{C9}$$

The resultant reflected ray is given by

$$\hat{v} = \sin\theta\sin(\phi - 2\alpha)\hat{x} - \sin\theta\cos(\phi - 2\alpha)\hat{y} - (\cos\theta)\hat{z}, \tag{C10}$$

which is a skew-line vector on the same hyperbolic envelope as $\hat{u}$ pointed in the direction of the waist with an endpoint at the point of reflection.

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