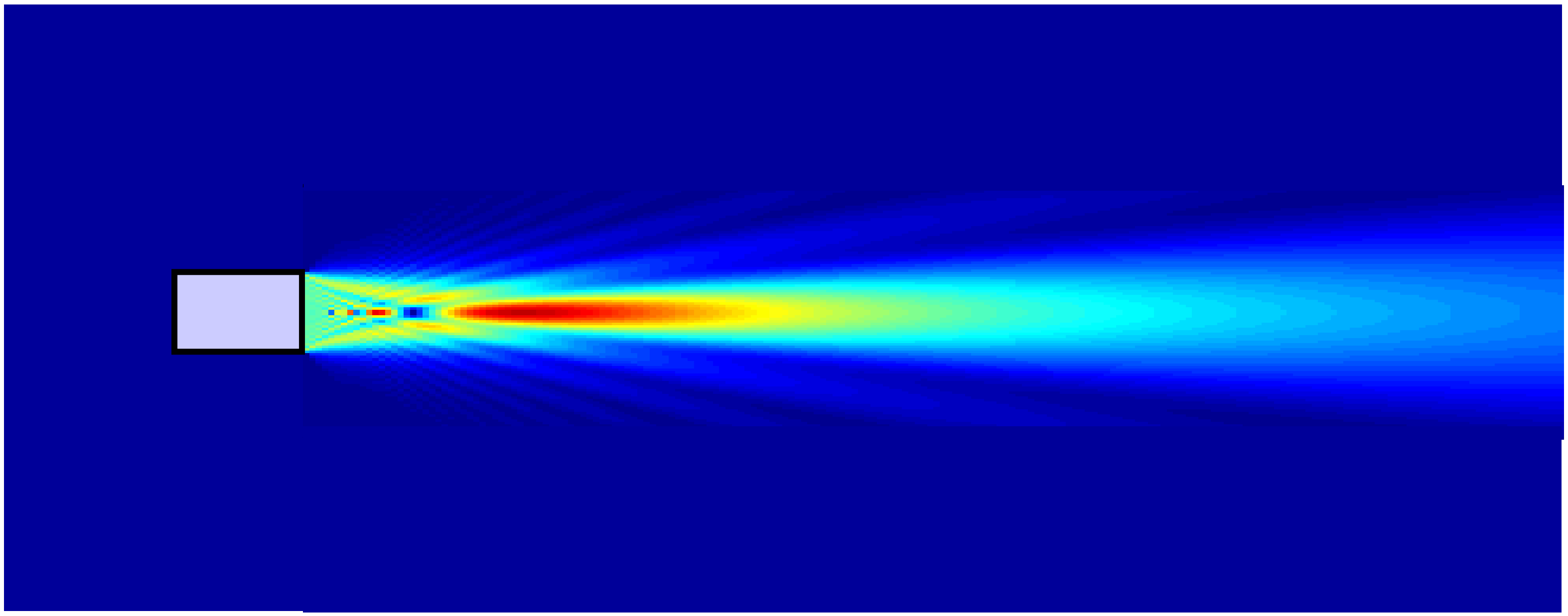
The Paraxial Approximation

Learning Objectives

Introduction to the paraxial approximation and its importance in beam modeling

The Paraxial Approximation

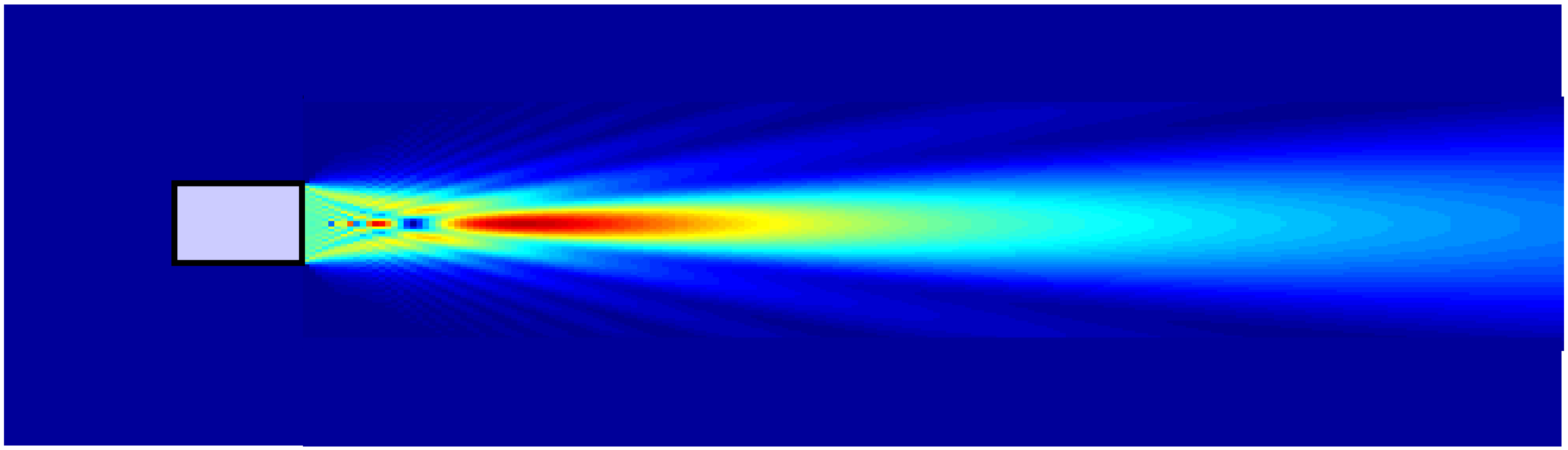
A transducer generates a complex beam of sound. Modeling that complexity is a challenging task.



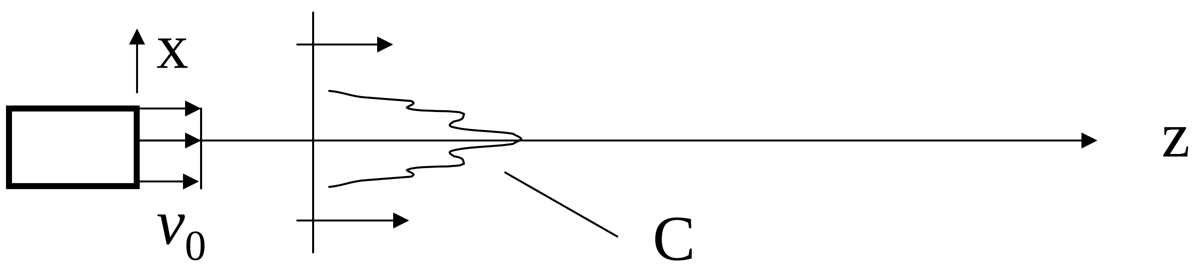
The modeling computational burden can be reduced considerably by introducing approximations. The paraxial approximation is one of the most useful of those approximations.

The Paraxial Approximation

The paraxial approximation models the transducer beam as a quasi-plane wave where most of the sound is propagating in



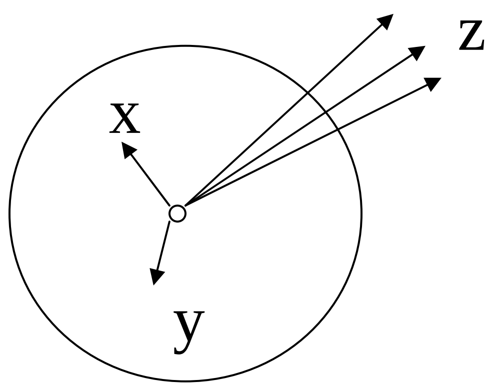
a given direction with an amplitude profile described by a <u>diffraction correction term</u>, C(x,y,z,ω).



$$p = \rho c v_0 C\left(x, y, z, \omega\right) \exp\left(ikz - i\omega t\right)$$

The Paraxial Approximation

To illustrate the paraxial approximation in a simple setting consider a spherical wave. Suppose that we are only interested in the spherical wave field in the vicinity of a particular direction which we will take as the z-axis (i.e. x,y <<z)



$$p = p_0 \frac{r_0 \exp(ikr)}{r}$$

$$v_z \cong v_r \cong \frac{p_0}{\rho c} \frac{r_0 \exp(ikr)}{r}$$

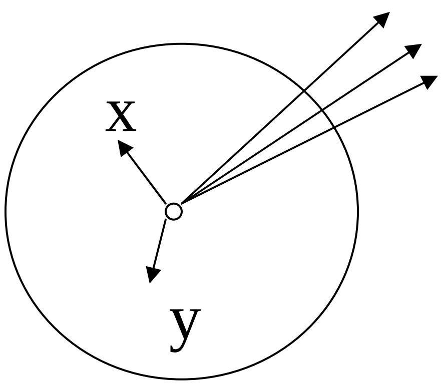
The Paraxial Approximation

Then, since
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$= z\sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}$$
$$\cong z + \frac{\rho^2}{2z}$$

where $\rho = \sqrt{x^2 + y^2}$

we find
$$p = p_0 \frac{r_0 \exp\left(\dfrac{ik\rho^2}{2z}\right)}{z} \exp(ikz) = p_0 C(x, y, z; \omega) \exp(ikz)$$
$$v_z = \frac{p_0}{\rho c} \frac{r_0 \exp\left(\dfrac{ik\rho^2}{2z}\right)}{z} \exp(ikz) = \frac{p_0}{\rho c} C(x, y, z; \omega) \exp(ikz)$$

The Paraxial Approximation



$$p = p_0 \frac{r_0 \exp\left(\frac{ik\rho^2}{2z}\right)}{z} \exp(ikz) = p_0 C(x, y, z; \omega) \exp(ikz)$$

$$v_z = \frac{p_0}{\rho c} \frac{r_0 \exp\left(\frac{ik\rho^2}{2z}\right)}{z} \exp(ikz) = \frac{p_0}{\rho c} C(x, y, z; \omega) \exp(ikz)$$

Note that in obtaining this diffraction correction term we only approximated the amplitude part of the spherical wave to first order (r ~ z) while we approximated the phase to second order. This is because terms neglected in the phase must not only be small with respect to the terms retained but also must be small with respect to 2π if they are to be negligible.

The Paraxial Approximation

Consider the wave equation

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}=0$$

Let $\quad p=P\left(x,y,z,\omega\right)\exp\left[ikz-i\omega t\right]$

Then $\quad \frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}+2ik\frac{\partial P}{\partial z}=0$

If we assume $\quad \left|\frac{\partial^2 P}{\partial z^2}\right| << \left|2ik\frac{\partial P}{\partial z}\right|,\left|\frac{\partial^2 P}{\partial x^2}\right|,\left|\frac{\partial^2 P}{\partial y^2}\right|$

$$\Rightarrow \quad \frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+2ik\frac{\partial P}{\partial z}=0$$

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + 2ik\frac{\partial P}{\partial z} = 0$$

paraxial wave equation

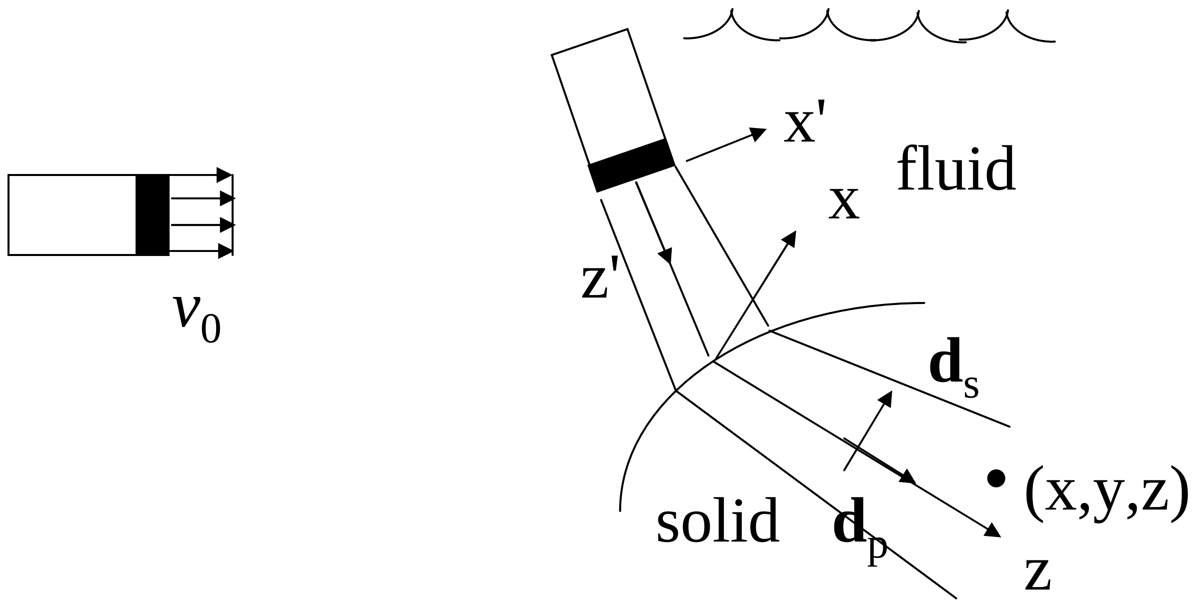
Our paraxial approximation for a spherical wave satisfies this paraxial equation exactly:

$$P = \frac{p_0 r_0}{z}\exp\left(\frac{ik\rho^2}{2z}\right)$$

There are other solutions, such as Gaussian waves that also satisfy this equation and form important building blocks for modeling ultrasonic transducer radiation in complex problems

The Paraxial Approximation

The paraxial approximation allows one to obtain diffraction correction terms for many practical testing setups



$k_1 = \omega / c_{p1}$

$k_2 = \omega / c_{p2}$
or $k_2 = \omega / c_{s2}$

$$\mathbf{v}\left(x, y, z, z', \omega\right) = v_0 T_{12} \mathbf{d}\, C\left(x, y, z, z', \omega\right) \exp\left(ik_1 z' + ik_2 z\right)$$

velocity

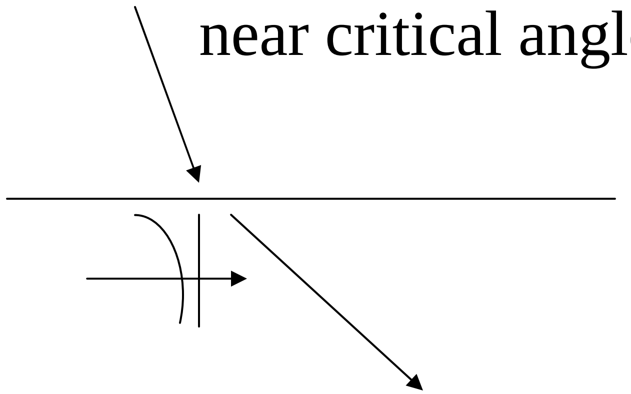
plane wave transmission coefficient

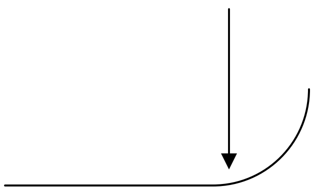
polarization $\mathbf{d}_p$ or $\mathbf{d}_s$

diffraction correction

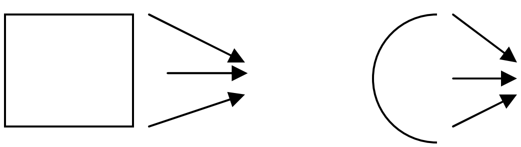
Limitations of Beam Models based on the Paraxial Approximation

near critical angles

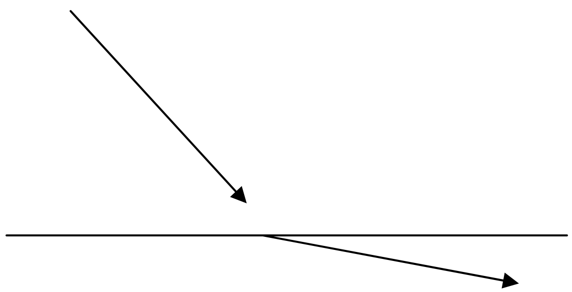




cases where the surface curvature
varies rapidly over the beam width



at locations too close to
a plane transducer or for
too tightly focused transducers



at high refracted angles